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Static Properties of the Deuteron**

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# Bethe-Salpeter Amplitudes and Static Properties of the Deuteron

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## Abstract

Extended calculations of the deuteron's static properties, based on the numerical solution of the Bethe-Salpeter equation, are presented. A formalism is developed, which provides a comparative analysis of the covariant amplitudes in various representations and nonrelativistic wave functions. The magnetic and quadrupole moments of the deuteron are calculated in the Bethe-Salpeter formalism and the rôle of relativistic corrections is discussed.

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# 1 Introduction

A theory applicable for studying nuclear phenomena, involving high energies or momentum transfers of a few GeV or larger, should be formulated in relativistically invariant manner. A traditional approach to processes with nuclei, based on the nonrelativistic Schrödinger wave functions, is not adequate if a large momentum transfer "sneaks" into the nuclear amplitudes, and the corresponding nucleon momentum  $p$  becomes large, say  $p \geq m$  ( $m$  is the nucleon mass). One can extend the usage of nonrelativistic wave functions by incorporating successively the relativistic corrections  $\sim (p/m)^n$ , however, it might eventually fail at some value of  $p$ . On the other hand, the nonrelativistic approach was for some time the only one which allowed for a detailed description of the static properties of the nuclei and low and intermediate energy nuclear reactions as well.

In the recent two decades, extensive studies of few-nucleon systems have been performed within Lorentz invariant models [1, 2, 3, 4, 5]. The success of these elaborate studies allows one to conclude that the covariant approach has now the capability to replace, at least for few-nucleon systems, the approaches relying on nonrelativistic wave functions [6]. Most of the phenomenological success in the relativistic treatment of few-nucleon systems has been achieved within such models which are based on a covariant meson-nucleon theory and corresponding dynamical equations [1, 2, 3]. In these models, the satisfactory results have been obtained for the nucleon-nucleon (NN) scattering, the properties of the lightest nuclei, various electromagnetic and hadronic interactions with nuclei, and some advance have been achieved for many-body nuclear systems (see e.g. discussions and further references in [3]).

The deuteron, as the simplest nuclear system, is an appealing object to be described by the models invented in the realm of nuclear physics. There is a fair amount of experimental information available about the deuteron's properties themselves and reactions with the deuteron. More interesting and precise data is expected after the start of the exciting research program at CEBAF. Therefore, there is a possibility to compare exact theoretical results with the experimental data in a clear way, not dimmed by extra effects, such as the "more-than-two-body" phenomena.

Still, the relativistic approach to the deuteron is not as popular as the one utilizing nonrelativistic wave functions [7, 8]. There are seemingly two main reasons for this. *First*, the deuteron, as any other nucleus, is essentially a nonrelativistic system, since it is composed of weakly bound massive nucleons. The bulk of the static properties of such a system obviously can be fitted in the nonrelativistic approach by adjusting the phenomenological potential or the wave function. Besides, the experimental data for the reactions with the deuteron is also mainly available in the nonrelativistic domain. *Second*, the relativistic models, especially those based on field theory, are technically more difficult and have a more sophisticated physical interpretation than the nonrelativistic approaches. Both these reasons, together, define the typical pattern for the attempts devoted to promote the consistent relativistic description of the nuclei. The corresponding works are usually highly specified for the particular reactions or kinematic domains where the advantage of the covariant

approach can be explicitly displayed. They are often filled with technical details uncommon for that part of the scientific audience which is not directly involved in this research direction. That is why this is so important to have simple and intuitively clear interpretations of the relativistic calculations, and an explicit systematic method to compare the relativistic and nonrelativistic results.

In the present work we are going to analyze the extended calculations of the static properties of the deuteron utilizing the Bethe-Salpeter (BS) amplitudes which are recently computed numerically [9]. The main goal of our paper is to contribute to the development of the physical intuition for understanding the relativistic calculations and their comparison to the nonrelativistic calculations. Our basic idea is to compute the observable densities of various charges (e.g., vector and axial-vector charges) in both the relativistic and the nonrelativistic formalisms and to use these densities as tools to compare relativistic amplitudes and nonrelativistic wave functions, which can not be rigorously interrelated otherwise. In doing so we pursue, in some sense, the goals opposite to the ones we outlined above as typical for the approach within the covariant description of the deuteron. Another goal of our paper is to fill some gap in the literature by giving explicit expressions relating the BS amplitudes in different representations, which will help to compare the relativistic amplitudes computed in different models.

We calculate here the magnetic and quadrupole moments of the deuteron within the Bethe-Salpeter formalism. The investigation of these static characteristics of the deuteron is still an important topic in nuclear physics. In the nonrelativistic models it gives the direct information about the tensor components in the nucleon - nucleon interaction and the magnitude of the  $D$  wave probability in the deuteron. However, there is an essential problem in fitting the experimental values of the quadrupole and magnetic moments with the same  $D$  wave probability in the nonrelativistic calculations (cf. ref. [10] and references therein). The efforts, aiming to solve this difficulty, go in two main directions, namely calculating the corrections of the meson exchange currents [11, 12, 13] and taking into account the relativistic effects [1, 2, 3, 14, 15, 16, 17]. In the conventional approach, the mesonic degrees of freedom and relativistic effects are treated as corrections to the nonrelativistic potential theory. It is found that, by adding these effects to the quadrupole moment, a satisfactory description of the data may be achieved for a broad range of different potentials [10], while the magnetic moment shows a stronger sensitivity to the model calculations of the meson exchange currents. Moreover, the consistency of such calculations is not at all clear. For this reason a comprehensive covariant investigation has its own right. A prominent feature of the relativistic consideration within the Bethe-Salpeter formalism is that the meson exchange effects due to pair creation currents is taken into account consistently [10, 18, 19], so that the essential part of the mentioned effects may be estimated in a self consistent way.

The general approach to calculate the static characteristic of the deuteron within the BS formalism has been elaborated by several authors since some time (see, for instance, refs. [3, 14, 15, 16, 20]) and numerical estimates have been performed. However, explicit calculations have been done within additional approximations for the solution of the BS

equation, e.g., for a separable interaction and by disregarding the negative energy states [15], or with one nucleon on mass-shell [3], or from a general point of view with adjusting the probability of the  $P$  states in order to fit simultaneously both the quadrupole and magnetic moments [16] (for this goal one needs an anomalously large pseudo-probability of the  $P$  waves, say  $\sim 1.5\%$ ). In the present paper we perform a covariant calculation of the quadrupole and magnetic moments of the deuteron within the exact solution of the BS equation and avoid additional approximations to the problem.

Our present investigation is also partially motivated by the renewed interest in the experimental investigation of the nucleon and deuteron spin-dependent structure functions at low momentum transfer  $Q^2$  [21]. This interest is connected to the study of the  $Q^2$  evolution of the Gerasimov-Drell-Hearn sum rule [22], which relates the spin-dependent structure functions of the targets to their the magnetic moments. For instance, only a correct description of the deuteron magnetic and quadrupole moments will assure a reliable extraction of the information about the neutron structure function from the deuteron data.

Our paper is organized as follows. In section II the basic covariant formulae for the electromagnetic current and static moments of the deuteron are presented. In section III the general definitions of the Bethe-Salpeter amplitudes for the deuteron are given in different representations and their symmetry properties are studied in detail. The transformation matrix relating the amplitudes in different representations is determined. The relativistic amplitudes are compared to the nonrelativistic wave functions, using the calculated observables, e.g., the vector and axial charge densities. In section IV the covariant formulae for the magnetic and quadrupole moments are derived in the Breit frame. The effects of the Lorentz deformation and dependence of the amplitude on the relative energy of the two nucleons in the deuteron are explicitly taken into account. The terms corresponding to the nonrelativistic expressions for the moments are determined in explicit form and the relativistic corrections are computed. The sections V and VI contain conclusions and the summary, respectively.

## 2 Relativistic kinematics of the electromagnetic current

The definition of the quadrupole moment  $Q_D$  and the magnetic moment  $\mu_D$  of the deuteron appears most transparent if one starts with the famous Rosenbluth formula (cf. [24]) for the elastic scattering of electrons off the deuteron,  $e + D \rightarrow e' + D'$ ,

$$\left. \frac{d\sigma}{d\Omega} \right|_{Lab} = \left. \frac{d\sigma}{d\Omega} \right|_{Mott} \left( A(q^2) + B(q^2) \tan^2 \frac{\theta}{2} \right) \quad (1)$$

with the following decomposition of the electromagnetic formfactors

$$A(q^2) = F_C^2(q^2) + \frac{8}{9}\eta^2 F_Q^2(q^2) + \frac{2}{3}\eta F_M^2(q^2), \quad (2)$$

$$B(q^2) = \frac{4}{3}\eta(1 + \eta) F_M^2(q^2), \quad (3)$$

where  $\eta = -q^2/4M_D^2$  and  $M_D$  is the deuteron mass.  $Q^2 = -q^2$  denotes the momentum transfer. Then the quadrupole and magnetic moments of the deuteron are defined via the normalization conditions for the charge ( $F_C$ ), quadrupole ( $F_Q$ ) and magnetic ( $F_M$ ) formfactors at vanishing momentum transfer  $q^2 = 0$

$$F_C(0) = 1, \quad F_Q(0) = M_D^2 Q_D, \quad F_M(0) = \mu_D \frac{M_D}{m}. \quad (4)$$

The general form of the deuteron electromagnetic current, which is invariant under Lorentz and time-reverse transformations, is given by

$$\langle P', \lambda' | J_\mu | P, \lambda \rangle = -\frac{e}{2M_D} \varepsilon_\rho^*(\mathbf{P}', \lambda') J_\mu^{\rho\sigma} \varepsilon_\sigma(\mathbf{P}, \lambda), \quad (5)$$

where  $\varepsilon^*(\mathbf{P}', \lambda')$  and  $\varepsilon(\mathbf{P}, \lambda)$  are the polarization four vectors of the initial and final deuteron states; greek sub/superscripts denote Lorentz indices to be moved with the Minkowski metric  $g_{\mu\nu}$ ;  $\mathbf{P}$  and  $\lambda$  stand for the three momentum and helicity of the deuteron. The covariant normalization of the current reads

$$\lim_{q^2 \rightarrow 0} \langle P', \lambda' | J_\mu | P, \lambda \rangle = e \frac{P_\mu}{M_D} \delta_{\lambda', \lambda}. \quad (6)$$

The matrix element  $J_{\rho\sigma}^\mu$  can be expanded in terms of the scalar formfactors in the form

$$J_{\rho\sigma}^\mu = (P' + P)^\mu \left( g_{\rho\sigma} F_1(q^2) - \frac{q_\rho q_\sigma}{2M_D^2} F_2(q^2) \right) + (g_\rho^\mu q_\sigma - g_\sigma^\mu q_\rho) G_1(q^2). \quad (7)$$

The scalar formfactors  $F_{1,2}$  and  $G_1$  are related to the formfactors  $F_{C,Q,M}$  by (cf. [23])

$$F_C(q^2) = F_1(q^2) + \frac{2}{3}\eta[F_1(q^2) + (1 + \eta)F_2(q^2) - G_1(q^2)], \quad (8)$$

$$F_Q(q^2) = F_1(q^2) + (1 + \eta)F_2(q^2) - G_1(q^2), \quad (9)$$

$$F_M(q^2) = G_1(q^2). \quad (10)$$

In the nonrelativistic impulse approximation these deuteron formfactors read

$$F_C(q^2) = (G_E^p(q^2) + G_E^n(q^2)) C_E(q^2), \quad (11)$$

$$F_Q(q^2) = (G_E^p(q^2) + G_E^n(q^2)) C_Q(q^2), \quad (12)$$

$$F_M(q^2) = \frac{M_D}{m} \left[ (G_E^p(q^2) + G_E^n(q^2)) C_S(q^2) + \frac{1}{2}(G_M^p(q^2) + G_M^n(q^2)) C_L(q^2) \right], \quad (13)$$

where  $G_E^{(p,n)}(q^2)$  ( $G_M^{(p,n)}(q^2)$ ) are the electric (magnetic) nucleon formfactors and the invariant functions  $C(q^2)$  are defined by

$$C_E(q^2) = \int_0^\infty dr (u^2 + w^2) j_0\left(\frac{qr}{2}\right), \quad C_E(0) = 1, \quad (14)$$

$$C_Q(q^2) = \frac{3\sqrt{2}}{2\eta} \int_0^\infty dr \left( uw - \frac{w^2}{2\sqrt{2}} \right) j_2\left(\frac{qr}{2}\right), \quad C_Q(0) = M_D^2 Q_D, \quad (15)$$

$$C_L(q^2) = \frac{3}{2} \int_0^\infty dr w^2 \left[ j_0\left(\frac{qr}{2}\right) + j_2\left(\frac{qr}{2}\right) \right], \quad C_L(0) = \frac{3}{2} P_D, \quad (16)$$

$$C_S(q^2) = \int_0^\infty dr \left( u^2 - \frac{w^2}{2} \right) j_0\left(\frac{qr}{2}\right) + \int_0^\infty dr \left( \frac{uw}{\sqrt{2}} + \frac{w^2}{2} \right) j_2\left(\frac{qr}{2}\right),$$

$$C_S(0) = 1 - \frac{3}{2} P_D. \quad (17)$$

Here  $j_i$  is the modified Bessel function of  $i$ -th order,  $u$  and  $w$  represent the  $S$  and  $D$  waves of the nonrelativistic deuteron wave function, and  $P_D$  is the weight of the  $D$  wave in the deuteron wave function.

To calculate the formfactors  $F_{C,Q,M}$  within the Bethe-Salpeter formalism one has to express the current (5) in terms of the BS amplitudes and, then, to extract the coefficients of different Lorentz structures given by eq. (7). Taking the limit  $q^2 \rightarrow 0$ , the static moments can be also obtained. Apparently, these calculations can be done in any particular reference frame. For example, the Breit frame is especially convenient for such type of calculations. The Breit frame is defined by the four momenta components of the deuteron

$$P_0 = P'_0 = E, \quad \mathbf{P} = -\frac{\mathbf{q}}{2}, \quad \mathbf{P}' = \frac{\mathbf{q}}{2}. \quad (18)$$

Choosing  $\mathbf{q}$  along the positive  $z$  axis and contracting  $J_{\rho\sigma}^\mu$  with the polarization vectors  $\varepsilon(\mathbf{P}', \lambda')$  and  $\varepsilon(\mathbf{P}, \lambda)$ , which obey

$$\varepsilon(\mathbf{P}', 1) = \varepsilon(\mathbf{P}, 1) = -\frac{1}{\sqrt{2}}(0, 1, i, 0), \quad (19)$$

$$\varepsilon(\mathbf{P}', -1) = \varepsilon(\mathbf{P}, -1) = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \quad (20)$$

$$\varepsilon(\mathbf{P}, 0) = (-\sqrt{\eta}, 0, 0, \sqrt{1+\eta}), \quad \varepsilon(\mathbf{P}', 0) = (\sqrt{\eta}, 0, 0, \sqrt{1+\eta}), \quad (21)$$

one arrives at expressions for the matrix elements of the deuteron electromagnetic current in terms of the formfactors:

$$\langle P', \lambda' | J^0 | P, \lambda \rangle = e \sqrt{1+\eta} (F_1 \delta_{\lambda\lambda'} + 2\eta [F_1 + (1+\eta)F_2 - G_1] \delta_{\lambda',0} \delta_{\lambda,0}), \quad (22)$$

$$\langle P', \lambda' | J^x | P, \lambda \rangle = e \frac{\sqrt{\eta}}{2} \sqrt{1+\eta} G_1 (\delta_{\lambda',\lambda+1} - \delta_{\lambda',\lambda-1}), \quad (23)$$

$$\langle P', \lambda' | J^y | P, \lambda \rangle = -ie \frac{\sqrt{\eta}}{2} \sqrt{1+\eta} G_1 (\delta_{\lambda',\lambda+1} + \delta_{\lambda',\lambda-1}), \quad (24)$$

$$\langle P', \lambda' | J^z | P, \lambda \rangle = 0. \quad (25)$$

Thus the magnetic and quadrupole formfactors of the deuteron are recovered by

$$\mu_D = \frac{m}{M_D} \sqrt{2} \lim_{\eta \rightarrow 0} \frac{\langle P', \lambda' = 1 | J^x | P, \lambda = 0 \rangle}{\sqrt{\eta} \sqrt{1+\eta}}, \quad (26)$$

$$Q_D = \frac{1}{M_D^2} \lim_{\eta \rightarrow 0} \frac{\langle P', \lambda' = 0 | J^0 | P, \lambda = 0 \rangle - \langle P', \lambda' = 1 | J^0 | P, \lambda = 1 \rangle}{2\eta \sqrt{1+\eta}}. \quad (27)$$



Equations (22)-(27) are the basic relations providing the calculations of the electromagnetic characteristics of the deuteron. In practice, one needs to define explicitly the operator  $J_\mu$  of the electromagnetic current and calculate its matrix elements with the deuteron states  $|P, \lambda\rangle$ .

### 3 The bound state wave function

#### 3.1 General definitions

Using the technique presented in ref. [9], the BS equation for a bound state in ladder approximation can be written in the form

$$K(p_0, \mathbf{p}) \chi(p; P) + \sum_B \frac{\lambda_B}{4i\pi^3} \int d^4 p' \frac{\Lambda(p_1) \Gamma_B \chi(p'; P) \Gamma_B \Lambda(p_2)}{(p - p')^2 - \mu_B^2} = 0, \quad (28)$$

$$K(p_0, \mathbf{p}) \equiv \left( E_p^2 - p_0^2 - \frac{1}{4} M_D^2 \right)^2 - p_0^2 M_D^2, \quad (29)$$

where  $\chi(p; P)$  is the BS amplitude for the deuteron in the matrix representation [9];  $\Lambda(p_i) = \hat{p}_i - m$ ;  $p = (p_0, \mathbf{p})_i$  is the four momentum of the  $i$ -th nucleon in the deuteron expressed in terms of relative four-momenta  $p$  or  $p'$  and the center-of-mass (c.m.) momentum  $P = (M_D, \mathbf{0})$ :  $p_{1,2} = P/2 \pm p$ ;  $B$  enumerates the exchanged mesons;  $\mu_B$  is the mass of the meson;  $\Gamma_B$  is the interaction vertex between the nucleon and the corresponding boson  $B$ , and  $\lambda_B \equiv g_B^2/4\pi$  with  $g_B$  being the coupling constant. We use here the short hand notation  $\hat{p} \equiv p^\mu \gamma_\mu$  for contractions with Dirac matrices  $\gamma_\mu$ .

Since the BS amplitude  $\chi$  and its adjoint  $\bar{\chi}$  satisfy the homogeneous BS equation they are determined up to an arbitrary constant which is fixed by an additional normalization condition. In the ladder approximation the normalization constant may be fixed by computing the matrix element of the electromagnetic current at  $q^2 = 0$ , i.e.,

$$\int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \{ \bar{\chi}(p; P) \gamma_\mu \chi(p; P) (m - \hat{p}_1) \} = 2P_\mu. \quad (30)$$

The normalization condition (30) coincides with the one used in ref. [14].

The BS amplitude is a  $(4 \times 4)$  matrix in the spinor space, and consequently the BS equation (28) possesses this matrix structure as well. To solve this matrix equations one can utilize a decomposition of the BS amplitude over a complete set of  $(4 \times 4)$  matrices and solves a system of coupled equations for the coefficients of such a decomposition. The choice of the representation of the matrices depends on the concrete attacked problem. Certainly, different representations are related by linear transformations, and it is straightforward (but cumbersome) to transform results from one representation to another one. In our opinion, to solve the BS equation and to compute matrix elements of the deuteron observables (as for instance eq. (30)), a convenient way is to decompose the amplitude in terms of the complete set of Dirac matrices, which form the Clifford algebra (for more details cf. ref. [9]). By

exploiting the parity invariance of the BS amplitude

$$\mathcal{P} \chi_D(p_0, \mathbf{p}) = \eta_P \gamma_0 \chi_D(p_0, -\mathbf{p}) \gamma_0, \quad (31)$$

it may be written for the deuteron, which has positive parity eigenvalues  $\eta_P = 1$ , as

$$\chi_D(p; P) = \gamma_5 P + \gamma^5 \gamma^0 A^0 - (\boldsymbol{\gamma} \cdot \vec{V}) - \gamma_5 (\boldsymbol{\gamma} \cdot \vec{A}) - 2i \gamma^0 (\boldsymbol{\gamma} \cdot \vec{T}^0) - 2 \gamma^0 \gamma_5 (\boldsymbol{\gamma} \cdot \vec{T}), \quad (32)$$

with pseudo-scalar ( $P, A^0$ ), axial ( $\vec{A}$ ) and vector ( $\vec{T}^0, \vec{T}, \vec{V}$ ) functions depending only upon the relative four momentum  $p$  in the c.m. frame. The angular dependence of the state with spin  $J = 1$  and its projection  $\mathcal{M}$  owing to the rotational invariance of eq. (28) is expressed in terms of the spherical and vector spherical harmonics. For example, when denoting  $\mathbf{X} = (P, A^0)$  and  $\vec{\mathbf{X}} = (\vec{A}, \vec{T}^0, \vec{T}, \vec{V})$ , we may write

$$\chi(p_0, \mathbf{p}) = \mathbf{X}_1(p_0, |\mathbf{p}|) Y_{1\mathcal{M}}(\Omega_p), \quad \vec{\chi}(p_0, \mathbf{p}) = \sum_{L=0,1,2} \mathbf{X}_L(p_0, |\mathbf{p}|) \mathbf{Y}_{1\mathcal{M}}^L(\Omega_p). \quad (33)$$

The corresponding equations for the radial functions can be found by a partial wave decomposition of the kernel in eq. (28) and by carrying out the angular integration. An example of the system of coupled equation for the radial amplitudes in the case of one exchanged scalar boson is given in ref. [9]. In what follows the notation for the radial amplitudes are kept as in eq. (32) with the lower index indicating the value of the angular momentum  $L$  in eq. (33).

### 3.2 The transformation properties of the partial amplitudes

Due to the parity invariance, eq. (31), only eight radial components are relevant to describe the deuteron amplitude, namely

$$P_1, \quad A_1^0, \quad A_0, \quad A_2, \quad V_1, \quad T_1^0, \quad T_0, \quad T_2. \quad (34)$$

Analyzing the behavior of the amplitude under the symmetry transformations, one can establish the properties of the components (34). The invariance of the BS equation under the time-reversal operation  $\mathcal{T}$

$$\mathcal{T} \chi_{\mathcal{M}}^D(p_0, \mathbf{p}) = \gamma^1 \gamma^3 \chi_{\mathcal{M}}^{D*}(p_0, -\mathbf{p}) \gamma^1 \gamma^3 \quad (35)$$

and the complex conjugation  $\mathcal{K}$

$$\mathcal{K} \chi_{\mathcal{M}}^D(p) = (-1)^{\mathcal{M}} \chi_{-\mathcal{M}}^D(p) \quad (36)$$

imply that the seven partial amplitudes  $P_1, V_1, A_1^0, A_{0,2}, T_{0,2}$  are real functions, while the amplitude  $T_1^0$  is purely imaginary, i.e.,  $T_1^{0*} = -T_1^0$ .

The Pauli principle implies that the amplitude  $\chi_D(p)$  changes the sign if two nucleons are interchanged, i.e.,

$$\chi_D(p_0, \mathbf{p}) = -\chi_D^{\dagger}(-p_0, -\mathbf{p}). \quad (37)$$

From eqs. (37) and (36) follows that  $A_1^0$  and  $T_1^0$  are odd functions with respect to the operation  $\Pi(p_0 \rightarrow -p_0)$

$$\Pi A_1^0(p_0, \mathbf{p}) = -A_1^0(p_0, \mathbf{p}), \quad \Pi T_1^0(p_0, \mathbf{p}) = -T_1^0(p_0, \mathbf{p}), \quad (38)$$

and the remaining six amplitudes are even functions of  $p_0$ . This symmetry property is useful for the classification of the amplitude according to two-nucleon states with a given relative energy, i.e., the  $\rho$  spin classification.

Table 1 summarizes the properties of the partial BS amplitudes in the representation (32) under the symmetry transformations.

### 3.3 Observables

Relying on the symmetry properties of the partial amplitudes, defined by eq. (32), the BS equation (28) has been solved numerically [9] for the deuteron at rest by performing a Wick rotation  $p_0 \rightarrow ip_4$ . In our present calculations we include six meson exchanges of  $\pi$ ,  $\omega$ ,  $\rho$ ,  $\sigma$ ,  $\eta$  and  $\delta$ , which describe the effective  $NN$  forces. The set of the meson parameters, such as masses, coupling constants and cut-off formfactors, employed here is the same as in ref. [14], obtained from a fit of the phase-shifts of the  $NN$  scattering and the binding energy of the deuteron.

The BS amplitude does not have a direct probabilistic interpretation as the Schrödinger wave function. Moreover, there is no simple way to compare these two objects describing the same system, namely the deuteron. In order to make a comparison possible, we can compute the same matrix elements of observables in two the approaches and compare these observables.

For example, the  $\mu = 0$  component of the normalization condition (30) in the rest frame of the deuteron, due to  $\langle D | \bar{N}(0) \gamma^0 N(0) | D \rangle = 2M_D$ , is simply a charge of the deuteron associated with the vector current. In the Wick rotated system and in terms of the partial amplitudes (34) it reads

$$\begin{aligned} M_D = & 2 \int \frac{dp_4 d|\mathbf{p}| |\mathbf{p}|^2}{(2\pi)^4} \left\{ -M_D \left( P_1^2 + A_1^{02} + 4T_1^{02} + V_1^2 \right) \right. \\ & + (2m_N - M_D) \left( X_0^{+2} + X_2^{+2} \right) - (2m_N + M_D) \left( X_0^{-2} + X_2^{-2} \right) \\ & + \frac{2\sqrt{2}|\mathbf{p}|}{\sqrt{3}} P_1 \left( X_0^+ - \sqrt{2}X_2^+ + X_0^- - \sqrt{2}X_2^- \right) \\ & \left. - \frac{2\sqrt{2}|\mathbf{p}|}{\sqrt{3}} V_1 \left( \sqrt{2}X_0^+ + X_2^+ - \sqrt{2}X_0^- - X_2^- \right) \right\}, \quad (39) \end{aligned}$$

where  $\vec{X}^\pm \equiv \sqrt{2}(\vec{T} \pm \vec{A}/2)$ . Now we define the charge density  $\rho_{\text{ch}}(|\mathbf{p}|)$  as

$$\frac{1}{2M_D} \langle D | \bar{N}(0) \gamma^0 N(0) | D \rangle = \int \frac{dp_4 d|\mathbf{p}| |\mathbf{p}|^2}{(2\pi)^4} \rho_{\text{ch}}(p_4, |\mathbf{p}|), \quad (40)$$

$$\rho_{\text{ch}}(|\mathbf{P}|) \equiv \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \rho_{\text{ch}}(p_4, |\mathbf{P}|). \quad (41)$$

This already may be compared with the corresponding nonrelativistic analogue, i.e., the square of the deuteron wave function in the momentum space, which is proportional to  $u^2(p) + w^2(p)$ .

In the same manner also the nucleon spin-density may be defined as density of the axial charge

$$\frac{1}{2M_D} \langle D | \bar{N}(0) \gamma_5 \gamma^0 N(0) | D \rangle = \int \frac{dp_4 d|\mathbf{p}| |\mathbf{p}|^2}{(2\pi)^4} \rho_{\text{spin}}(p_4, |\mathbf{p}|), \quad (42)$$

$$\rho_{\text{spin}}(|\mathbf{P}|) \equiv \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \rho_{\text{spin}}(p_4, |\mathbf{P}|). \quad (43)$$

In the nonrelativistic limit this density reflects the contribution of the  $D$  wave admixture in the deuteron, which is proportional to  $u^2(p) - \frac{1}{2}w^2(p)$ .

Results of numerical calculation of the defined densities together with a comparison with their nonrelativistic counterparts obtained with the Bonn and Paris potentials are presented in figs. 1 and 2. All curves exhibit qualitatively similar shapes and are identical in the nonrelativistic region  $|\mathbf{p}| \leq 0.5$  GeV/c. If the momentum  $|\mathbf{p}|$  increases, the deviations of the relativistic results from the nonrelativistic ones becomes more significant, but still too small to be attributed to relativistic effects. Rather it is compatible with the model differences. Particular attention is to be paid to the fig. 2, where the spin density is depicted. This function is rather sensitive to the internal spin-orbital structure of the deuteron. The fact, that the "elementary oscillations" of the spin density in the potential models are reproduced by the solution of the BS equation, might be interpreted as the relativistic structure of the deuteron which is governed by the nucleon interaction in states with a positive energy and  $L = 0, 2$ , i.e., by  ${}^3S_1$  and  ${}^3D_1$  configurations. Therefore, in spite of the quadratic forms of the partial amplitudes, which are not diagonal in eqs. (39) and (43), one can define the relativistic analogue of the probability of the  $D$  wave admixture in the deuteron. Carrying out the  $|p|$  integration in eq. (43) and equating the result to  $(1 - 3/2 P_D)$  we find  $P_D \approx 5\%$  (cf. [14]), which is compatible with the probabilities of the Bonn ( $P_D = 4.3$  [8]) and Paris ( $P_D = 5.9$  [7]) potential models.

### 3.4 The BS amplitude in different representations

To have a closer analogue with the nonrelativistic consideration it is convenient to use another basis set of matrices in the decomposition of the BS amplitude. In the literature the two-spinor basis [26] is frequently used, which means an outer product of two spinors, representing solutions of the free Dirac equation with positive and negative energies. This basis is labeled by the relative momentum  $\mathbf{p}$ , the helicities  $\lambda_i$  and the energy spin  $\rho_i$  of the particles [14],

sometimes also called  $(J, \lambda_1, \lambda_2, \rho_1, \rho_2)$  representation. In this case one usually adopts for the partial amplitudes the spectroscopic notation  $^{2S+1}L_J^{\rho_1, \rho_2}$ , i.e.,

$${}^3S_1^{++}, {}^3S_1^{--}, {}^3D_1^{++}, {}^3D_1^{--}, {}^1P_1^{+-}, {}^1P_1^{-+}, {}^3P_1^{+-}, {}^3P_1^{-+}. \quad (44)$$

Sometimes it is more convenient to change from the  $(J, \lambda_1, \lambda_2, \rho_1, \rho_2)$  representation to the representation  $(J, L, S, \rho)$  where  $\rho$  is the projection of the total energy spin of the system. In this case the notation of the components is as follows

$$Y^T \equiv (v_s^o, v_t^e, v_s^e, v_t^o, u^+, u^-, w^+, w^-), \quad (45)$$

where  $u, v, w$  correspond to  $L = 0, 1, 2$ , respectively, and  $o$  or  $e$  mean the odd or even parity relative to the  $\rho$  spin function; the lower indices  $s$  and  $t$  denote the singlet and triplet spin configurations, respectively. According to eqs. (37) and (36), the amplitudes  $v_s^o, v_t^e$  are odd and  $v_s^e, v_t^o$  are even functions of  $p_0$ . The partial amplitudes in the basis (44), (45) are of a more familiar form and show a more transparent physical meaning since they may be compared with the deuteron states in the nonrelativistic limit. It is intuitively clear (see also figs. 1 and 2) that the two nucleons in the deuteron are mainly in states with  $L = 0, 2$  and with positive energy so that one may expect that the probability of states with negative energies and  $L = 1$  in eqs. (44) - (45) is much smaller in comparison with the probability for the  ${}^3S_1^{++}$  and  ${}^3D_1^{++}$  (or  $u^+$  and  $w^+$ ) configurations. Moreover, it can be shown that the waves  ${}^3S_1^{++}$  and  ${}^3D_1^{++}$  directly correspond to the  $S$  and  $D$  waves in the deuteron, while those with the negative energy vanish in the nonrelativistic limit.

The partial amplitudes (44) are defined through the following decomposition of the BS amplitude

$$\chi_D(p_0, \mathbf{p}) = \sum_{\alpha} \phi_{\alpha}(p_0, |\mathbf{p}|) \mathcal{V}_{\mathcal{M}}^{\alpha}(\mathbf{p}), \quad (46)$$

where  $\alpha = \{J, L, S, \rho_1, \rho_2\}$  labels different states of the system;  $\phi_{\alpha}$  denotes the partial amplitudes in eq. (45), and  $\mathcal{V}_{\mathcal{M}}^{\alpha}(\mathbf{p})$  are the spin-angular functions

$$\mathcal{V}_{\mathcal{M}}^{\alpha}(\mathbf{p}) = i^L \sum_{s_1 s_2 m} (Lm Ss | J\mathcal{M}) \left( \frac{1}{2} s_1 \frac{1}{2} s_2 | Ss \right) Y_{Lm}(\hat{\mathbf{p}}) U_{s_1}^{\rho_1}(\mathbf{p}) U_{s_2}^{\rho_2}(-\mathbf{p}). \quad (47)$$

In eq. (47) the quantities  $U_s^{\rho}(\mathbf{p})$  are the free nucleon spinors; the explicit matrix form for the spin-angular functions  $\mathcal{V}_{\mathcal{M}}^{\alpha}(\mathbf{p})$  is given in the Appendix I.

In order to establish a connection between the representation (32) and the spinor basis (44)-(46) we represent the Dirac matrices in eq. (32) as a direct product of Pauli matrices of the nucleon spin  $\boldsymbol{\sigma}$  and the  $\rho$  spin

$$\chi_D(p) = \rho^1 \otimes [\hat{I}P - 2i(\boldsymbol{\sigma} \cdot \vec{T}_0)] + i\rho^2 \otimes [\hat{I}A^0 + (\boldsymbol{\sigma} \cdot \vec{V})] + \rho^3 \otimes (\boldsymbol{\sigma} \cdot \vec{A}) + 2I \otimes (\boldsymbol{\sigma} \cdot \vec{T}). \quad (48)$$

The last two terms in eq. (48) may be rewritten as

$$\rho^3 \otimes (\boldsymbol{\sigma} \cdot \vec{A}) + 2I \otimes (\boldsymbol{\sigma} \cdot \vec{T}) = \frac{1}{\sqrt{2}} (\hat{I} + \rho^3) \otimes (\boldsymbol{\sigma} \cdot \vec{X}^+) + \frac{1}{\sqrt{2}} (\hat{I} - \rho^3) \otimes (\boldsymbol{\sigma} \cdot \vec{X}^-). \quad (49)$$

Then eqs. (48) and (49), together with the symmetry properties of our partial amplitudes listed in the Table 1, show that the desired relation between the two representations appears as follows:

$$\begin{aligned} {}^3S_1^{++} &\sim X_0^+, \quad {}^3S_1^{--} \sim X_0^-, \quad {}^3D_1^{++} \sim X_2^+, \quad {}^3D_1^{--} \sim X_2^-, \\ {}^3P_1^e &\sim T_1^0, \quad {}^3P_1^o \sim V_1, \quad {}^1P_1^e \sim P_1, \quad {}^1P_1^o \sim A_1^0. \end{aligned}$$

The relation between (34) and (45) can be established exactly. The components being odd in the relative energy  $v_s^o$ ,  $v_t^e$  and  $A_0^1$ ,  $T_0^1$  are related directly to each other via

$$v_s^o = -iA_0^1, \quad v_t^e = 2T_0^1, \quad (50)$$

whereas the remaining six components are connected via linear combinations. By representing these amplitudes as six-component vectors,  $\tilde{Y}^T = (v_s^e, v_t^o, u^+, u^-, w^+, w^-)$  and  $\Psi^T = (P_1, V_1, X_0^+, X_0^-, X_2^+, X_2^-)$ , the transition from  $\tilde{Y}$  to  $\Psi$  is provided by a unitary transformation  $\tilde{Y} = U\Psi$  (with  $\det(U) = -1$ , and  $UU^T = 1$ ) with the following explicit form of the transition matrix

$$U = \frac{\zeta}{2\sqrt{1+\zeta^2}} \times \begin{pmatrix} -\frac{2}{\zeta} & 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{2}{\zeta} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & -\frac{2}{\sqrt{3}} & \frac{1+\sqrt{1+\zeta^2}}{\zeta} & \frac{1-\sqrt{1+\zeta^2}}{3\zeta} & 0 & \frac{2\sqrt{2}}{3} \frac{1-\sqrt{1+\zeta^2}}{\zeta} \\ -\sqrt{\frac{2}{3}} & -\frac{2}{\sqrt{3}} & \frac{1-\sqrt{1+\zeta^2}}{3\zeta} & \frac{1+\sqrt{1+\zeta^2}}{\zeta} & \frac{2\sqrt{2}}{3} \frac{1-\sqrt{1+\zeta^2}}{\zeta} & 0 \\ \frac{2}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & -\frac{2\sqrt{2}}{3} \frac{1-\sqrt{1+\zeta^2}}{\zeta} & -\frac{1+\sqrt{1+\zeta^2}}{\zeta} & \frac{1-\sqrt{1+\zeta^2}}{3\zeta} \\ -\frac{2}{\sqrt{3}} & \sqrt{\frac{2}{3}} & -\frac{2\sqrt{2}}{3} \frac{1-\sqrt{1+\zeta^2}}{\zeta} & 0 & \frac{1-\sqrt{1+\zeta^2}}{3\zeta} & -\frac{1+\sqrt{1+\zeta^2}}{\zeta} \end{pmatrix}, \quad (51)$$

with  $\zeta = |\mathbf{p}|/m$ . In the nonrelativistic limit, where  $\zeta \ll 1$ , the matrix  $U$  becomes diagonal

$$U = \text{diag}(-1, 1, 1, 1, -1, -1), \quad (52)$$

and our representation coincides with the one in the spinor basis. In what follows all formulae will be derived in terms of the partial amplitudes (44) or (45), nevertheless the numerical calculations are performed with our solutions (34) by utilizing eqs. (50) and (51).

Coming back to the normalization condition it is easy to show that eq. (39) may be transformed to a diagonal form

$$\frac{2}{M_D} \int \frac{dp_4 d|\mathbf{p}| |\mathbf{p}|^2}{(2\pi)^4} (Y^+(p_4, |\mathbf{p}|), \hat{\omega} Y(p_4, |\mathbf{p}|)) = 1, \quad (53)$$

which is exactly the normalization condition used in ref. [14]. In eq. (53)  $Y$  denotes the eight component vector (45), and  $\hat{\omega}$  is a diagonal matrix

$$\hat{\omega} = -\text{diag}(M_D, M_D, M_D, M_D, M_D - 2E_p, 2E_p + M_D, M_D - 2E_p, 2E_p + M_D), \quad (54)$$

so that the integrand in eq. (53) consists of a sum of quadratic terms of radial functions  $Y_\alpha$  weighted with  $\omega_\alpha$ . Therefore each term, after integration, may be interpreted as pseudo-probability of finding the corresponding relativistic state in the deuteron. The result of our numerical calculations of the pseudo-probabilities is presented in Table 2. It is seen that an admixture of the negative-energy amplitudes affects the contribution of the positive-energy states. The appearance of the negative contributions of waves with negative  $\rho$  spin is not a surprise; it follows from the physical meaning of the normalization condition according to that the contribution of each term in eq. (53) is the effective baryon charge in the corresponding state. The pseudo-probabilities of  $S$  and  $D$  waves (see Table 2) are close to the corresponding probabilities obtained in the nonrelativistic Bonn and Paris potentials, as expected, since the deuteron is essentially a nonrelativistic system.

To investigate the behavior of the partial amplitudes and their nonrelativistic limits, we employ once more the normalization integral (30), now however in the form of eq. (53). Then, similar to eqs. (41) and (43), we define the following functions  $\psi^\alpha$  depending upon  $|\mathbf{p}|$  by

$$\psi^\alpha(|\mathbf{p}|) = \sqrt{2 \int d p_4 \omega_\alpha |Y_\alpha(p_4, |\mathbf{p}|)|^2 M_D^{-1}}. \quad (55)$$

Thus  $\psi^\alpha$  may be regarded as the absolute value of the relativistic wave function of the deuteron in the state  $\alpha$  (for instance,  $\alpha = 5$  corresponds to a  ${}^3S_1^{++}$  configuration,  $\alpha = 7$  to  ${}^3D_1^{++}$  etc., cf. eq. (45)).

Figures (3) and (4) display the behavior of the relativistic wave functions  $\psi_0^+$  and  $\psi_2^+$  (solid lines) versus the relative momentum  $|\mathbf{p}|$  in comparison with the nonrelativistic  $S$  and  $D$  waves. We conclude that with an accuracy of model ambiguities in the nonrelativistic calculations (given here by the difference between Paris and Bonn wave functions, i.e., the dashed lines in figs. 3 and 4) the large relativistic components are close to their nonrelativistic analogues up to  $|\mathbf{p}| \sim m$ . However, there is a distinctive difference in the shape of the  $D$  waves in the two approaches. Namely, the nonrelativistic functions change the sign in the region  $|\mathbf{p}| \sim m$ , whereas the BS component does not do so (cf. the solid line labeled as BS-I in fig. 4). To understand this we tentatively introduce an auxiliary definition of the relativistic  $D$  wave which is just the difference between the integrand in the normalization condition and the contribution of the  ${}^3S_1^{++}$  component, i.e., we introduce in the definition of the  $D$  wave the contribution of all the negative energy states:  $\tilde{\psi}_2^+ \sim \sqrt{w^{+2} + w^{-2} + u^{-2} + \dots}$ . In this case only two wave functions  $\psi_0^+$  and  $\tilde{\psi}_2^+$  determine the normalization of the BS amplitude, and the correspondence with the nonrelativistic limit becomes one to one. In fig. 4 the function  $\tilde{\psi}_2^+$  is labeled by BS-II, and it is seen that it displays a minimum in the same region as the nonrelativistic functions, i.e., it has the same shape as the nonrelativistic  $D$  wave. One observes that the nonrelativistic  $D$  wave already mimics relativistic effects, so that in calculations of relativistic corrections to the nonrelativistic approaches an overestimate of the magnitude of such corrections may occur. For completeness, in fig. 5 we present the wave functions for  $L = 1$ ; since the waves  $u^-$  and  $w^-$  are negligibly small, even in comparison with the waves  $L = 1$ , they are not presented here.

### 3.5 The vertex functions

In studying the nonrelativistic correspondence of the solutions of the BS equation it is convenient to work with the BS vertices  $G(p; P)$  defined by

$$\chi(p; P) = \frac{(\hat{p}_1 + m) G(p; P) (\hat{p}_2 + m)}{(p_1^2 - m^2)(p_2^2 - m^2)}. \quad (56)$$

From eqs. (46) and (56) it is possible to find a decomposition for the vertex  $G(p; P)$ . In doing so, one introduces the two four-vectors of on-mass-shell particles corresponding to the Dirac spinors in eq. (47), i.e.,

$$k_1 = (E_p, \mathbf{p}), \quad k_2 = (E_p, -\mathbf{p}), \quad E_p = \sqrt{\mathbf{p}^2 + m^2}, \quad p = (p_0, \mathbf{p}). \quad (57)$$

Then in eq. (56) the inverse propagator of the nucleons may be represented in terms of the vectors  $k_{1,2}$  by

$$\begin{aligned} S^{-1}(1) &\equiv \frac{\hat{P}}{2} + \hat{p} - m = \frac{1}{2E_p} \left[ (\hat{k}_1 - m) S_-^{-1}(1) + (\hat{k}_2 + m) S_+^{-1}(1) \right], \\ S^{-1}(2) &\equiv \frac{\hat{P}}{2} - \hat{p} - m = \frac{1}{2E_p} \left[ (\hat{k}_2 - m) S_-^{-1}(2) + (\hat{k}_1 + m) S_+^{-1}(2) \right], \end{aligned} \quad (58)$$

where

$$S_{\pm}(1) = \left( \frac{M_D}{2} + p_0 \mp E_p \right)^{-1}, \quad S_{\pm}(2) = \left( \frac{M_D}{2} - p_0 \mp E_p \right)^{-1}. \quad (59)$$

Because of

$$\begin{aligned} S^{-1}(1) U_s^{\rho_1}(\mathbf{p}) &= \text{sign}(\rho_1) S_{\rho_1}(1) U_s^{\rho_1}(-\mathbf{p}), \\ S^{-1}(2) U_s^{\rho_2}(-\mathbf{p}) &= \text{sign}(\rho_2) S_{\rho_2}(2) U_s^{\rho_2}(\mathbf{p}), \end{aligned} \quad (60)$$

the decomposition of  $G(p; P)$  reads

$$G_{\mathcal{M}}(p; P) = \sum_{\alpha} G^{\alpha}(p_0, |\mathbf{p}|) \mathcal{V}_{\mathcal{M}}^{\alpha}(-\mathbf{p}), \quad (61)$$

hence the partial amplitudes and the vertex functions are interrelated via the following simple expression

$$Y^{\alpha}(p_0, |\mathbf{p}|) = S_{\rho_1}(1) S_{\rho_2}(2) G^{\alpha}(p_0, |\mathbf{p}|). \quad (62)$$

The relation eq. (62) implies that the BS amplitudes (46) have sharp maxima around  $p_0 = 0$ , while the behavior of the partial vertices  $G^{\alpha}(p_0, |\mathbf{p}|)$  is predicted to appear as smooth functions of the relative energy (see also ref. [14]).

The behavior of the vertex functions is shown in figs. 6 and 7 for the configurations  ${}^3S_1^{++}$  and  ${}^3D_1^{++}$  as functions of the relative energy  $p_0$  and momentum  $|\mathbf{p}|$  in the Wick-rotated system. One observes that the dependence of the vertex functions upon the relative energy is weak, hence one may expect that the nonrelativistic and relativistic vertices at  $p_0 = 0$  have similar structures as functions of  $|\mathbf{p}|$ . From this observation and eq. (62) we



establish another relation between the BS amplitudes and nonrelativistic wave functions. Below, as an example, we show how one can obtain the relativistic wave function for the  ${}^3S_1^{++}$  configuration from the BS amplitude. The energy dependence of the component  $u^+$  is factorized into two parts, namely a dependence on the scalar propagators (59) and a vertex function. Then, using the smoothness of the vertex as function of  $p_0$  we replace it by its value at  $p_0 = 0$  multiplied with a smooth function of  $p_0$ , i.e.,

$$u^+(p_0, |\mathbf{p}|) = \frac{G^+(p_0, |\mathbf{p}|)}{[(\frac{M_D}{2} - E_p)^2 - p_0^2]} = \frac{G^+(0, |\mathbf{p}|) \xi(p_0, |\mathbf{p}|)}{[(\frac{M_D}{2} - E_p)^2 - p_0^2]} \quad (63)$$

with  $\xi(0, |\mathbf{p}|) = 1$ , where the dimensionless function  $\xi(p_0, |\mathbf{p}|)$  reflects the energy dependence of the vertex function. In view of the smooth behavior of the vertices as function on  $p_0$ , one may replace this function by a constant,  $\xi(0, |\mathbf{p}|) \approx \xi_0$  with  $\xi_0 \sim 1$ . Then in the normalization integral eq. (53) the integration over the relative energy may be carried out explicitly and the remaining part corresponds to the square of the nonrelativistic wave function, i.e., we define the nonrelativistic limit of the BS amplitude  $u^+$  by

$$\psi_0(|\mathbf{p}|) = \xi_0 u^+(0, |\mathbf{p}|) \frac{(M_D - 2E_p)}{4\sqrt{M_D}}. \quad (64)$$

Similar definitions, using eqs. (53), (54), (59) and (62), are valid for other waves. The generalized relativistic  $S$  and  $D$  waves in this manner are displayed in figs. 8 and 9. These figures should be compared with figs. 3 and 4, which display the modulus (that is without the sign) of the wave functions, whereas figs. 8 and 9 rely on the absolute values. One can consider this as a new way of finding the nonrelativistic analogues of the BS amplitudes. The actual calculations have been performed with  $\xi_0 = 1$ . A comparison with the corresponding nonrelativistic wave functions at  $|\mathbf{p}| \rightarrow 0$  shows that, by choosing the parameter  $\xi_0 = 1$ , we slightly overestimate (by about 10%) the relativistic functions (see figs. 6 and 7). It is worth stressing that in our solution of the BS equation the relativistic  $D$  wave does not change its sign in the interval up to  $|\mathbf{p}| \sim 1.5$  GeV/c. This is the most essential difference between the relativistic and nonrelativistic approaches in this region. Therefore, one can expect that the relativistic corrections to physical quantities in the deuteron up to  $|\mathbf{p}| \sim 1$  GeV/c are relatively small; to distinguish them one should either compute observables which are known experimentally with a very high precision and sensitive to the spin structure, or find special processes where the large components are suppressed and only the states with negative energies are relevant.

## 4 The static characteristics of the deuteron

Let us calculate now the static moments of the deuteron in the BS formalism. The conserved electromagnetic current of the deuteron (5) in terms of the BS amplitude is given by

$$\langle P', \lambda' | J_\mu | P, \lambda \rangle = -ieN_D \int d^4p \text{Tr} \{ \bar{\chi}_{\lambda'}(\not{p}'; P') \Gamma_\mu(q) \chi_\lambda(\not{p}; P) S_F(\hat{p}_2)^{-1} \}, \quad (65)$$

where  $S_F(p_2) = \hat{p}_2 + m$ ,  $p' = p + q/2$ ,  $P' = P + q$ ,  $N_D = 1/(2\pi)^4/2M_D$ . The quantity  $\Gamma_\mu$  is the photon-nucleon electromagnetic vertex, which is assumed to be of the on-mass-shell form

$$\Gamma_\mu(q) = \gamma_\mu F_1^s(q^2) - \frac{\kappa}{2m} \sigma_{\mu\nu} q^\nu F_2^s(q^2), \quad (66)$$

where  $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ , and  $F_i^s$  are isoscalar Dirac (Pauli) formfactors of the nucleon with  $F_1(0) = F_2(0) = 1/2$ ,  $\kappa = \mu_p + \mu_n - 1$ , and  $\mu_{p,n}$  are the proton and neutron anomalous magnetic moments in units of the nuclear magneton  $e/(2m)$ . The gauge invariance of the electromagnetic current in the ladder approximation has been proven in ref. [14] (see also [25]).

Now we have to interrelate the expression for the static moments (26) and (27), which are determined in the Breit frame, and the BS amplitudes, which are numerically obtained in the rest frame of the deuteron. This relation is given by the general transformation rules

$$\chi_\lambda(p; P) = \Lambda(\mathcal{L}) \chi_\lambda(\mathcal{L}^{-1}p; P_{c.m.}) \Lambda^{-1}(\mathcal{L}), \quad (67)$$

$$\bar{\chi}_\lambda(p'; P') = \Lambda^{-1}(\mathcal{L}) \bar{\chi}_\lambda(\mathcal{L}p'; P_{c.m.}) \Lambda(\mathcal{L}), \quad (68)$$

$$\Lambda^{-1}(\mathcal{L}) S_F(\frac{1}{2}P - p)^{-1} \Lambda(\mathcal{L}) = S_F(\frac{1}{2}P_{c.m.} - \mathcal{L}^{-1}p)^{-1}, \quad (69)$$

where  $\Lambda$  is the operator for spin- $\frac{1}{2}$  particles corresponding to the Lorentz transformation  $P = \mathcal{L}P_{c.m.}$ ,  $P' = \mathcal{L}^{-1}P_{c.m.}$ ,

$$\Lambda(\mathcal{L}) = \frac{M_D + \hat{P}\gamma_0}{\sqrt{2M_D(E + M_D)}} \quad (70)$$

with the corresponding Lorentz transformation matrix  $\mathcal{L}$

$$\mathcal{L} = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \quad (71)$$

The direction of the boost is supposed to be parallel to  $q_Z$ . Then, after the Lorentz transformation of the integrand in eq. (65), the matrix element takes the form

$$\langle P', \lambda' | J_\mu | P, \lambda \rangle = -ieN_D \int d^4p \text{Tr} \left\{ \bar{\chi}_{\lambda'}(p'; P_{c.m.}) \tilde{\Gamma}_\mu(q^2) \chi_\lambda(p; P_{c.m.}) S_F(\frac{1}{2}P_{c.m.} - p)^{-1} [\Lambda^{-1}(\mathcal{L})]^2 \right\}, \quad (72)$$

where

$$\tilde{\Gamma}_\mu(q) = \Lambda(\mathcal{L}) \Gamma_\mu(q) \Lambda(\mathcal{L}) \quad (73)$$

and the variable  $p'$  is represented via  $p$  and  $q$  as

$$p' = \mathcal{L}p'^{(B)} = \mathcal{L}(p^{(B)} + \frac{1}{2}q) = \mathcal{L}^2 p + \frac{1}{2}\mathcal{L}q, \quad (74)$$

with components

$$p^{0'} = (1 + 2\eta)p_0 - 2\sqrt{\eta}\sqrt{1 + \eta}p^z - M_D\eta, \quad (75)$$

$$p^{x'} = p^x, \quad p^{y'} = p^y, \quad (76)$$

$$p^{z'} = (1 + 2\eta)p^z - 2\sqrt{\eta}\sqrt{1 + \eta}p^0 + M_D\sqrt{\eta}\sqrt{1 + \eta}. \quad (77)$$

Eq. (72) is the starting point in evaluating the static moments of the deuteron in the BS formalism. The main peculiarities of this matrix element, in comparison with the familiar nonrelativistic expression, come from the Lorentz transformation and from the relativistic nature of the BS amplitude itself and might be characterized by

- (i) effects of the negative-energy partial states (especially nondiagonal expectation values of the current between  ${}^3S_1^{++}$  and  ${}^1P_1^{(e),(o)}$ ,  ${}^3P_1^{(o),(e)}$  partial states),
- (ii) a dependence of the amplitude upon the relative energy  $p_0 \neq 0$ ; in studying the static characteristics of the deuteron this effect is called retardation in the BS amplitude,
- (iii) an effect of boosting to the internal space-time variable, that is the effect of  $\mathcal{L} \neq 1$ ,
- (iv) effects of the deformation of the BS amplitude concerning the booster  $\Lambda(\mathcal{L}) \neq 1$ .

In fact, in the matrix element (72) these boost effects reduce to a deformation of the photon-nucleon vertex eq. (73) and to corrections from  $[\Lambda^{-1}(\mathcal{L})]^2$ . In our case, i.e., as  $\eta \rightarrow 0$  (see eqs. (26) and (27)) for the eqs. (72) and (73) one may write

$$[\Lambda^{-1}(\mathcal{L})]^2 \simeq 1 + \sqrt{\eta}\gamma_0\gamma_3 + \frac{\eta}{2}, \quad (78)$$

$$\Lambda(\mathcal{L})\gamma_0\Lambda(\mathcal{L}) = \gamma_0 \quad (79)$$

$$\Lambda(\mathcal{L})\gamma_1\Lambda(\mathcal{L}) = \gamma_1[\Lambda(\mathcal{L})]^2 \quad (80)$$

$$\Lambda(\mathcal{L})\gamma_\alpha\hat{q}\Lambda(\mathcal{L}) = \gamma_\alpha\hat{q}, \quad (\alpha = 0, 1). \quad (81)$$

In what follows, the deviation of the quantity  $[\Lambda^{-1}(\mathcal{L})]^2$  from unity in the matrix element eq. (72) we call the effects of the Lorentz boost in the BS amplitude.

## 4.1 The quadrupole moment

### 4.1.1 General formulae

According to eqs. (27), (66) and (78)-(81) the result for the quadrupole momentum is presented as follows

$$\begin{aligned} Q_D &= \sum_{\alpha, \alpha'} \sum_{\rho, \rho'} \langle a'^{\rho'} | \hat{Q} | a^\rho \rangle \\ &= \sum_{\alpha, \alpha'} \sum_{\rho, \rho'} \left[ \langle a'^{\rho'} | \hat{Q}_C | a^\rho \rangle + \langle a'^{\rho'} | \hat{Q}_C^{LB} | a^\rho \rangle + \langle a'^{\rho'} | \hat{Q}_M | a^\rho \rangle + \langle a'^{\rho'} | \hat{Q}_M^{LB} | a^\rho \rangle \right], \quad (82) \end{aligned}$$

where the subscripts  $C$  and  $M$  mean the corresponding contribution of the charge and magnetic part of the photon-nucleon vertex (66), and the superscript  $LB$  is the contribution of the Lorentz boost  $[\Lambda(\mathcal{L})^{-1}]^2 - 1$ .

The corresponding matrix elements of the zeroth component of the deuteron electromagnetic current in the definition (27) take the form ( $\lambda = 0, 1$ )

$$J_0^{(\lambda,\lambda)}(P, P) = \frac{e}{2M_D} \int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_\lambda(p'; P) \gamma_0 \chi_\lambda(p; P) S_F\left(\frac{P}{2} - p\right)^{-1} \right\}, \quad (83)$$

$$J_0^{(\lambda,\lambda)}(P, P) = \sqrt{\eta} \frac{e}{2M_D} \int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_\lambda(p'; P) \gamma_0 \chi_\lambda(p; P) S_F\left(\frac{P}{2} - p\right)^{-1} \gamma_0 \gamma_3 \right\}, \quad (84)$$

$$J_0^{(\lambda,\lambda)}(P, P) = -\frac{e}{2M_D} \frac{\kappa}{4m_N} \times \int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_\lambda(p'; P) \Lambda(\mathcal{L}) (\gamma_0 \hat{q} - \hat{q} \gamma_0) \Lambda(\mathcal{L}) \chi_\lambda(p; P) S_F\left(\frac{P}{2} - p\right)^{-1} \right\}, \quad (85)$$

$$J_0^{(\lambda,\lambda)}(P, P) = -\frac{e}{2M_D} \frac{\kappa \sqrt{\eta}}{4m_N} \times \int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_\lambda(p'; P) (\gamma_0 \hat{q} - \hat{q} \gamma_0) \chi_\lambda(p; P) S_F\left(\frac{P}{2} - p\right)^{-1} \gamma_0 \gamma_3 \right\}. \quad (86)$$

$$\int \frac{d^4 p}{i(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_\lambda(p'; P) (\gamma_0 \hat{q} - \hat{q} \gamma_0) \chi_\lambda(p; P) S_F\left(\frac{P}{2} - p\right)^{-1} \gamma_0 \gamma_3 \right\}. \quad (87)$$

As next step the partial wave decomposition of eqs. (83) - (87) has to be performed. Then one expands the integrands in Taylor series around  $\eta = 0$  and carries out the limit  $\eta \rightarrow 0$ . It is clear that one has to keep corrections including  $O(\eta)$  in both the wave function  $\chi(p'_0, \mathbf{p}'; P)$  and the matrix  $\Lambda(\mathcal{L})$ .

This scheme of calculation allows to investigate separately the contribution of different relativistic effects mentioned above. The eqs. (84) and (87) are new contributions which account for the effect of the boosted photon-nucleon vertex. Moreover, also the Lorentz deformation effect of the BS amplitude is taken into account in these matrix elements through the relative momentum  $p'$ .

Obviously, the main contributions to the quadrupole moment come from the charge part  $\langle a^{\rho'} | Q_C | a^\rho \rangle$ , computed with the large  $S$  and  $D$  components of the BS amplitude. For these states, with  $\rho = \rho' = +1$ , one can recover the nonrelativistic formula for the quadrupole moment of the deuteron and separate the corrections due to the relativistic Fermi motion of the nucleons and the retardation in the relative energy

$$Q_D^C = \sum_{a, a' = S, D} \langle a'^+ | \hat{Q}_C | a^+ \rangle = Q_p^{(+,+)} + Q_{p_0}^{(+,+)}. \quad (88)$$

The two terms in r.h.s. of the eq. (88) reflect the existence of derivatives with respect to the momentum  $|\mathbf{p}|$  and the relative energy in the corresponding integrands

$$Q_p^{(+,+)} = -\frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \left( E_{\mathbf{p}} - \frac{M_D}{2} + p_0 \right) \times \left\{ \left( 1 - \frac{2p_0}{M_D} \right)^2 \left[ -\frac{1}{12} \frac{(E_{\mathbf{p}} - m)^2}{|\mathbf{p}|^2 E_{\mathbf{p}}^2} u^+(p_0, |\mathbf{p}|)^2 \right. \right. \\ \left. \left. - \frac{1}{120} \frac{14E_{\mathbf{p}}^4 + 5E_{\mathbf{p}}^2 m^2 - 3m^4 + 20E_{\mathbf{p}}^3 m}{|\mathbf{p}|^2 E_{\mathbf{p}}^4} w^+(p_0, |\mathbf{p}|)^2 \right] \right\} \quad (89)$$

$$\begin{aligned}
& + \frac{1}{10} w^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) + \frac{1}{20} w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial |\mathbf{p}|^2} w^+(p_0, |\mathbf{p}|) \\
& + \frac{\sqrt{2} 3m^4 - 4E_{\mathbf{p}}^4 + 5E_{\mathbf{p}}^2 m^2 + 5E_{\mathbf{p}}^3 m}{60 |\mathbf{p}|^2 E_{\mathbf{p}}^4} u^+(p_0, |\mathbf{p}|) w^+(p_0, |\mathbf{p}|) \\
& + \frac{\sqrt{2} 2E_{\mathbf{p}} + 3m}{20 E_{\mathbf{p}}} u^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) \\
& + \frac{\sqrt{2} 2E_{\mathbf{p}} - 3m}{20 E_{\mathbf{p}}} w^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} u^+(p_0, |\mathbf{p}|) \\
& + \frac{\sqrt{2}}{20} w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial |\mathbf{p}|^2} u^+(p_0, |\mathbf{p}|) + \frac{\sqrt{2}}{20} u^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial |\mathbf{p}|^2} w^+(p_0, |\mathbf{p}|) \Big] \\
& + \frac{1}{5} \frac{|\mathbf{p}|^2}{M_D^2} \left[ \frac{3}{2} \frac{1}{|\mathbf{p}|^2} w^+(p_0, |\mathbf{p}|)^2 + w^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) \right. \\
& + \frac{3\sqrt{2}}{|\mathbf{p}|^2} u^+(p_0, |\mathbf{p}|) w^+(p_0, |\mathbf{p}|) \\
& \left. + \sqrt{2} w^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} u^+(p_0, |\mathbf{p}|) + \sqrt{2} u^+(p_0, |\mathbf{p}|) \frac{1}{|\mathbf{p}|} \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) \right] \Big\}
\end{aligned}$$

and

$$\begin{aligned}
Q_{p_0}^{(+,+)} &= \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \frac{1}{5} \frac{|\mathbf{p}|^2}{M_D^2} (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) \times \\
& \left\{ -\sqrt{2} \left[ u^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0^2} w^+(p_0, |\mathbf{p}|) + w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0^2} u^+(p_0, |\mathbf{p}|) \right] \right. \\
& \left. - w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0^2} w^+(p_0, |\mathbf{p}|) \right\} \\
& + \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \frac{3}{10M_D} \left(1 - \frac{2p_0}{M_D}\right)^2 (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) \times \\
& \left\{ \sqrt{2} \left[ \left(1 + \frac{m}{E_{\mathbf{p}}}\right) u^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} w^+(p_0, |\mathbf{p}|) + \left(1 + \frac{m}{E_{\mathbf{p}}}\right) w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} u^+(p_0, |\mathbf{p}|) \right] \right. \\
& \left. + w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} w^+(p_0, |\mathbf{p}|) \right\} \\
& + \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \frac{|\mathbf{p}|}{5M_D} \left(1 - \frac{2p_0}{M_D}\right) (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) \times \\
& \left\{ \sqrt{2} \left[ u^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0 \partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) + w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0 \partial |\mathbf{p}|} u^+(p_0, |\mathbf{p}|) \right] \right. \\
& \left. + w^+(p_0, |\mathbf{p}|) \frac{\partial^2}{\partial p_0 \partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) \right\},
\end{aligned} \tag{90}$$

where  $u^+(p_0, |\mathbf{p}|)$  and  $w^+(p_0, |\mathbf{p}|)$  represent the radial function of the corresponding partial states  ${}^3S_1^{++}$  and  ${}^3D_1^{++}$ . In the nonrelativistic approximation,  $E_{\mathbf{p}} \rightarrow m_N$ ,  $p_0/M_D \rightarrow 0$ ,

eq. (89) yields

$$\begin{aligned}
Q_p^{(+,+)} \rightarrow & \\
& - \frac{e}{2M_D} \frac{1}{10\sqrt{2}} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \left\{ \left[ \left( -\frac{\partial^2}{\partial |\mathbf{p}|^2} + \frac{1}{|\mathbf{p}} \frac{\partial}{\partial |\mathbf{p}|} \right) u^+(p_0, |\mathbf{p}|) \right] w^+(p_0, |\mathbf{p}|) \right. \\
& - \left. \left[ \left( \frac{\partial^2}{\partial |\mathbf{p}|^2} + \frac{5}{|\mathbf{p}} \frac{\partial}{\partial |\mathbf{p}|} + \frac{3}{|\mathbf{p}|^2} \right) w^+(p_0, |\mathbf{p}|) \right] u^+(p_0, |\mathbf{p}|) \right\} (E_{\mathbf{p}} - \frac{M_D}{2}) \\
& + \frac{e}{2M_D} \frac{1}{20} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \left[ \left( \frac{\partial^2}{\partial |\mathbf{p}|^2} + \frac{2}{|\mathbf{p}} \frac{\partial}{\partial |\mathbf{p}|} - \frac{6}{|\mathbf{p}|^2} \right) w^+(p_0, |\mathbf{p}|) \right] \times \\
& w^+(p_0, |\mathbf{p}|) (E_{\mathbf{p}} - \frac{M_D}{2}). \tag{91}
\end{aligned}$$

The expression eq. (91) has not yet a "true" nonrelativistic form because of the integration over  $p_0$ . However, by utilizing eqs. (63), (64) with  $\xi_0 = 1$  and carrying out the  $p_0$  integration explicitly, the familiar nonrelativistic expression [27] for the quadrupole moment is reproduced exactly:

$$\begin{aligned}
Q_D = & -\frac{1}{20} \int \frac{d|\mathbf{p}|}{(2\pi)^3} \left\{ \sqrt{8} \left[ |\mathbf{p}|^2 \frac{d\psi_0(|\mathbf{p}|)}{d|\mathbf{p}|} \frac{d\psi_2(|\mathbf{p}|)}{d|\mathbf{p}|} + 3|\mathbf{p}| \psi_2(|\mathbf{p}|) \frac{d\psi_0(|\mathbf{p}|)}{d|\mathbf{p}|} \right] \right. \\
& \left. + |\mathbf{p}|^2 \left( \frac{d\psi_2(|\mathbf{p}|)}{d|\mathbf{p}|} \right)^2 + 6(\psi_2(|\mathbf{p}|))^2 \right\}, \tag{92}
\end{aligned}$$

where  $\psi_0(|\mathbf{p}|)$  and  $\psi_2(|\mathbf{p}|)$  are defined by eq. (64) and correspond to the nonrelativistic  $S$  and  $D$  components of the deuteron wave function (see, also figs. 8 and 9). As seen from eq. (92) the main contribution to the matrix element (89) is expected to come from the interference of the positive  $S$  and  $D$  states in the deuteron; the remaining terms with negative  $\rho$  spins are the contribution of the relativistic Fermi motion.

The second term  $Q_{p_0}^{(+,+)}$  in eq. (88) and the matrix element of the Lorentz boost operator (82) are of a pure relativistic nature and reflect the relativistic corrections to the quadrupole moment. For instance, for the positive states the corrections  $\hat{Q}_C^{LB}$ ,  $Q_{LB}^{(++)} \equiv \sum_{a,a'=S,D} \langle a'^+ | \hat{Q}_C^{LB} | a^+ \rangle$  are

$$\begin{aligned}
Q_{LB}^{(++)} = & \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) (1 - \frac{2p_0}{M_D}) \frac{1}{5M_D} \frac{1}{E_{\mathbf{p}}} \times \\
& \left\{ \frac{6E_{\mathbf{p}}^2 - 2mE_{\mathbf{p}} - m^2}{E_{\mathbf{p}}^2} \left[ \frac{1}{2} w^+(p_0, |\mathbf{p}|)^2 + \sqrt{2} u^+(p_0, |\mathbf{p}|) w^+(p_0, |\mathbf{p}|) \right] \right. \\
& + \sqrt{2} |\mathbf{p}| \left[ u^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) + w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial |\mathbf{p}|} u^+(p_0, |\mathbf{p}|) \right] \\
& \left. + |\mathbf{p}| w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial |\mathbf{p}|} w^+(p_0, |\mathbf{p}|) \right\} \tag{93}
\end{aligned}$$

$$\begin{aligned}
& -\frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) \frac{1}{5} \frac{|\mathbf{p}|^2}{M_D^2 E_{\mathbf{p}}} \times \\
& \left\{ \sqrt{2} \left[ u^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} w^+(p_0, |\mathbf{p}|) + w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} u^+(p_0, |\mathbf{p}|) \right] \right. \\
& \left. + w^+(p_0, |\mathbf{p}|) \frac{\partial}{\partial p_0} w^+(p_0, |\mathbf{p}|) \right\}. \tag{94}
\end{aligned}$$

After integration by part in eq. (93) and (94) one obtains

$$\begin{aligned}
Q_{LB}^{(++)} &= \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} (E_{\mathbf{p}} - \frac{M_D}{2} + p_0) \left(1 - \frac{2p_0}{M_D}\right) \frac{2}{5M_D} \frac{1}{E_{\mathbf{p}}} \times \\
& \left\{ \frac{2E_{\mathbf{p}}^2 - mE_{\mathbf{p}} - m^2}{E_{\mathbf{p}}^2} \left[ \sqrt{2} u^+(p_0, |\mathbf{p}|) w^+(p_0, |\mathbf{p}|) + \frac{1}{2} w^+(p_0, |\mathbf{p}|)^2 \right] \right\} \tag{95}
\end{aligned}$$

$$\begin{aligned}
& + \frac{e}{2M_D} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{dp_0 |\mathbf{p}|^2 d|\mathbf{p}|}{i(2\pi)^4} \frac{1}{5} \frac{|\mathbf{p}|^2}{M_D^2 E_{\mathbf{p}}} \left(1 - \frac{M_D}{E_{\mathbf{p}}}\right) \times \\
& \left\{ \sqrt{2} u^+(p_0, |\mathbf{p}|) w^+(p_0, |\mathbf{p}|) + \frac{1}{2} w^+(p_0, |\mathbf{p}|)^2 \right\}. \tag{96}
\end{aligned}$$

It is seen that the magnitude of this term is of order  $Q_{LB}^{(++)} \approx \langle \frac{|\mathbf{p}|^2}{M_D^2} \rangle Q_p^{(+,+)}$  and vanishes in the nonrelativistic limit. In order to achieve self-consistency in the nonrelativistic approach to the deuteron formfactors and electrodisintegration reactions, various relativistic corrections to the matrix elements must be taken into account, such as meson exchange currents and pair term contributions [13, 28, 29]. In the covariant description of the deuteron these effects are partially accounted for by calculating transitions between states with negative energies; the contribution of  $P$  states in the deuteron electromagnetic current corresponds to diagrams with nucleon-antinucleon pair creation in the old fashioned perturbation theory. Moreover, in ref. [18] it has been shown that, considering the deuteron electrodisintegration process within the light-front dynamics, beside the dominant contribution of expectation values with  $S$  and  $D$  waves, an extra matrix element with transitions between positive and negative energy states is relevant to describe the electrodisintegration amplitude. It has been also shown that the contribution of this extra component exactly reproduces the pair term corrections in the nonrelativistic limit. An investigation of the correspondence between the light-front dynamics approach and the BS amplitude has shown [19] that the extra component in ref. [18] may be imitated by transitions between a linear combination of the  $P$  waves and  $S$  or  $D$  waves. Hence in our calculation the pair terms are taken into account via calculations of off-diagonal expectation values of the relevant current between the  $S$  and  $P$  partial wave states (see also discussions in refs. [10, 16]). A more detailed analysis of the nonrelativistic limit of the expression for the quadrupole moment with keeping leading corrections  $\sim 1/m$  will be presented elsewhere.

### 4.1.2 Numerical results

The full expression for the quadrupole moment consists of a multitude of terms likewise eq. (89) with quadratic combinations of partial states and terms with second derivatives  $\partial^2/\partial|\mathbf{p}|^2$ ,  $\partial^2/\partial p_0^2$  and mixed ones  $\partial^2/\partial|\mathbf{p}|\partial p_0$  computed between different partial BS amplitudes. Their analytical form has been evaluated by an algebraic formula manipulation code. Numerical calculations have been performed by using our solutions of the BS equation for the partial amplitudes eq. (34) and the relations (50), (51). We find that the main contribution to the deuteron quadrupole moment gives the first term in eq. (82) and that the transitions between energy even-states dominate, i.e.,

$$Q_p^{++} = 0.2690 \text{ fm}^2. \quad (97)$$

This contribution is below the experimental data  $Q_D = (0.2859 \pm 0.0003) \text{ fm}^2$  [30] by about 6%, nevertheless it is larger than the usual nonrelativistic calculations. This is an understandable effect because of the specific feature of the solution of the BS equation for which the sum of the pseudo-probabilities of the positive  $S$  and  $D$  waves is larger than 1. In this context, since the pseudo-probabilities of the remaining configurations are negative, the transitions with  $P$  waves are expected to play an important rôle in studying the static characteristics of the deuteron. Particularly interesting is the calculation of the off-diagonal expectation value between the  $S$  and  $P$  partial states, which is predicted to replace the meson exchange contribution in nonrelativistic calculations [18, 19]. Indeed, our numerical result points to a significant contribution of the mentioned matrix elements in comparison with other nondiagonal transitions, namely

$$\langle u^+ | \hat{Q}_C | v_s^e \rangle = 0.0052 \text{ fm}^2, \quad \langle u^+ | \hat{Q}_C | v_t^e \rangle = -0.0027 \text{ fm}^2 \quad (98)$$

(for example, among other nondiagonal matrix elements the largest one is  $\langle w^+ | \hat{Q}_C | v_s^e \rangle = -0.00007 \text{ fm}^2$ ).

The part of the quadrupole moment with odd (diagonal and nondiagonal) expectation values gives a small negative contribution to the first term in eq. (82):  $\langle \hat{Q}_C \rangle_{\text{odd}} = -0.0007 \text{ fm}^2$ . Gathering together all the contributions we obtain  $Q_C = 0.2706 \text{ fm}^2$ .

An estimate of the corrections owing to the dependence on the relative energy, eq. (88), shows that they are rather small:  $Q_{p_0}^{(++)} = 0.0006 \text{ fm}^2$ .

The Lorentz boost corrections have been calculated and they are found to be negative. Their total contribution is  $Q_C^{LB} = -0.0029 \text{ fm}^2$  which, together with  $Q_C$ , gives the final result for the electric part of the quadrupole moment of the deuteron  $Q = 0.2683 \text{ fm}^2$ . An important moment should be stressed here. The contribution of the Lorentz boost terms with nondiagonal transitions between  $S$  and  $P$  waves are of the same order of magnitude as those in eq. (98) but of opposite sign

$$\langle v^+ | \hat{Q}_C^{LB} | v_s^e \rangle = -0.0053 \text{ fm}^2, \quad \langle v^+ | \hat{Q}_C^{LB} | v_t^e \rangle = 0.0027 \text{ fm}^2. \quad (99)$$

Equations (98) and (99) show that the contribution of pair creation terms in nonrelativistic calculations is predicted to be negligibly small for the quadrupole moment and confirm the qualitative results obtained in ref. [16].



Note that our classification of the matrix elements into the main part and Lorentz boost corrections (cf. eq. (82)) is rather conventional and does not reflect directly the contribution of relativistic effects. However, by using eqs. (88), (91) and (92) we may present our results in the form

$$Q_D = Q_{NR} + \delta Q_{rel} = (0.2690 - 0.0007) \text{ fm}^2, \quad (100)$$

where the nonrelativistic part  $Q_{NR}$  is determined by the large components of the BS amplitude and does not depend upon the derivatives with respect to the relative energy and upon the Lorentz boost effects;  $\delta Q_{rel}$  is the contribution of all the remaining terms and, obviously, is of a pure relativistic nature. It is seen that the relativistic corrections to the quadrupole moment are negative and reinforce the discrepancy, although their magnitude is rather small. A similar conclusion has been drawn in ref. [16] from a more qualitative analysis of the deuteron moment within the BS formalism.

Another source of the relativistic corrections is the contribution of the magnetic part of the effective current (82) which vanishes in the nonrelativistic limit. Our calculation show that its contribution to the quadrupole moment is negative too,  $\langle |\hat{Q}_M| \rangle = -0.0005 \text{ fm}^2$ , so that our final result for the deuteron quadrupole moment is  $Q_D = 0.2678 \text{ fm}^2$ , i.e., the discrepancy in  $Q_D$  is about 6%.

## 4.2 The magnetic moment

### 4.2.1 General formulae

According to eqs. (26), (65) and (78)-(81) the result for the magnetic moment can be written as

$$\mu_D = \mu_+ + \mu_{1-} + \mu_{2-} + \mu_{3-}, \quad (101)$$

where the matrix elements between states with positive energies in eq. (101) are labeled by the subscript +, and the subscript - means that the corresponding matrix element implements at least one wave with negative energy. The matrix elements  $\mu_{i-}$  reflect the relativistic corrections. In order to emphasize the nonrelativistic analogue of the magnetic moment in the expression for the  $\mu_+$  we subtract the corresponding nonrelativistic formula, and the remaining part we denote as  $R_+$ , which is the relativistic corrections due to the Fermi motion effects. Then the functions  $\mu_{...}$  can be represented by

$$\mu_+ = (\mu_p + \mu_n)(P_{u^+} + P_{w^+}) - \frac{3}{2}(\mu_p + \mu_n - \frac{1}{2})P_{w^+} + R_+, \quad (102)$$

$$\mu_{1-} = \frac{1}{2}(\mu_p + \mu_n)(P_{v_i^c} + P_{v_i^c}) + \frac{1}{4}(P_{v_i^c} + P_{v_i^c}) + \frac{1}{2}(P_{v_i^c} + P_{v_i^c}) + R_{1-}, \quad (103)$$

$$\mu_{2-} = -(\mu_p + \mu_n)P_{u^-} + P_{u^-} + \frac{1}{2}(\mu_p + \mu_n)P_{w^-} - \frac{5}{4}P_{w^-} + R_{2-}, \quad (104)$$

$$\mu_{3-} = \sum_{a,b} C^{a,b}, \quad (105)$$

where  $a = u^+, w^+, u^-, w^-$ ,  $b = v_s^e, v_s^o, v_t^e, v_t^o$ , and  $P_i$  are the pseudo-probabilities of the corresponding partial state. In eqs. (102)-(104) the diagonal expectation values between states with  $L = 0, 2, 1$  are written explicitly; the off-diagonal contributions are included in the terms  $R$  and  $\mu_{3-}$ , where

$$\begin{aligned}
R_+ &= -\frac{1}{3}(\mu_p + \mu_n - 1 + \frac{2m}{M})H_1^{u^+} - \frac{m}{M}H_2^{u^+} - \frac{m}{M}H_3^{u^+} - (1 - \frac{2m}{M})P_{u^+} \\
&\quad - \frac{1}{6}(\mu_p + \mu_n - 1 - \frac{4m}{M})H_1^{w^+} - \frac{m}{M}H_2^{w^+} - \frac{m}{M}H_3^{w^+} - \frac{1}{4}(1 - \frac{2m}{M})P_{w^+} \\
&\quad + \frac{\sqrt{2}}{3}(\mu_p + \mu_n - 1 - \frac{m}{M})H_1^{u^+, w^+}, \tag{106}
\end{aligned}$$

$$\begin{aligned}
R_{1-} &= -\frac{1}{2}(1 - \frac{2m}{M})(\mu_p + \mu_n + \frac{1}{2})(P_{v_t^e} + P_{v_t^o}) - \frac{1}{2}(1 - \frac{2m}{M})(P_{1P_1^e} + P_{1P_1^o}) \\
&\quad + \frac{2}{5}(2H_4^{v_t^e, v_t^e} - H_8^{v_t^e, v_t^e}) + \frac{1}{5}H_9^{v_t^e, v_t^e} - \frac{2}{5}H_{10}^{v_t^e, v_t^e} \\
&\quad + \frac{2}{5}(H_4^{v_s^e, v_s^e} + 2H_8^{v_s^e, v_s^e}) - \frac{2}{5}H_9^{v_s^e, v_s^e} + \frac{4}{5}H_{10}^{v_s^e, v_s^e} \\
&\quad + \sqrt{2}(\mu_p + \mu_n - 1 + \frac{4m^2}{M^2})H_5^{v_t^e, v_s^e} \\
&\quad + \sqrt{2}(\mu_p + \mu_n - 1)H_5^{v_t^o, v_s^e} - \frac{\sqrt{2}}{2}H_6^{v_t^o, v_s^e} - 2\sqrt{2}H_7^{v_t^o, v_s^e}, \tag{107}
\end{aligned}$$

$$\begin{aligned}
R_{2-} &= -\frac{1}{3}(\mu_p + \mu_n - 1 - \frac{2m}{M})H_1^{u^-} - \frac{m}{M}H_2^{u^-} + \frac{m}{M}H_3^{u^-} \\
&\quad - \frac{1}{6}(\mu_p + \mu_n - 1 + \frac{4m}{M})H_1^{w^-} - \frac{m}{M}H_2^{w^-} + \frac{m}{M}H_3^{w^-} \\
&\quad + \frac{3}{4}(1 - \frac{2m}{M})P_{w^-} + \frac{\sqrt{2}}{3}(\mu_p + \mu_n - 1 + \frac{m}{M})H_1^{u^-, w^-}. \tag{108}
\end{aligned}$$

The quantities  $C^{a,b}$  and  $H_i^{\alpha', \alpha}$  are given in the Appendix II. Now the nonrelativistic formula for the magnetic moment may be recovered exactly by rewriting the term  $\mu_+$  in the form

$$\mu_+ = \mu_{NR} + \Delta\mu_+, \tag{109}$$

where

$$\mu_{NR} = (\mu_p + \mu_n) - \frac{3}{2}(\mu_p + \mu_n - \frac{1}{2})P_D$$

reproduces the nonrelativistic formula, and the relativistic corrections due to the Fermi motion effects are

$$\Delta\mu_+ = R_+ - (\mu_p + \mu_n)(P_{u^-} + P_{w^-} + P_{v_s^e} + P_{v_s^o} + P_{v_t^e} + P_{v_t^o}). \tag{110}$$

Finally, the total contributions to the deuteron magnetic moment read

$$\mu_D = \mu_{NR} + \Delta\mu,$$

$$\Delta\mu = R_+ + \Delta\mu_- + \mu_{3-}, \quad (111)$$

$$\begin{aligned} \Delta\mu_- = & -(\mu_p + \mu_n) \left[ \frac{1}{2}(P_{v_i^e} + P_{v_i^o}) + (P_{v_i^e} + P_{v_i^o}) + 2P_{u^-} + \frac{1}{2}P_{w^-} \right] \\ & + \frac{1}{4}(P_{3P_1^e} + P_{v_i^o}) + \frac{1}{2}(P_{1P_1^e} + P_{1P_1^o}) + P_{u^-} - \frac{5}{4}P_{w^-} + R_{1-} + R_{2-}. \end{aligned} \quad (112)$$

#### 4.2.2 Numerical results

Explicit numerical calculations give for the total deuteron magnetic moment the value  $\mu_D = 0.856140 (e/2m)$  which differs from the experimentally known moment  $\mu_{exp} = (0.857406 \pm 10^{-6}) (e/2m)$  [16] by less than 0.15%. This result consists of the nonrelativistic contribution plus the following relativistic corrections:

(i) the main correction to the nonrelativistic value of the magnetic moment  $\mu_{NR} = 0.850718 (e/2m)$  that comes from the transitions between positive energy states and  $P$  states ( $v_s^e, v_s^o, v_i^e, v_i^o$ , cf. the term  $\mu_{3-}$  in eq. (111)); it gives  $\mu_{3-} = 6.099 \cdot 10^{-3} (e/2m)$  and contains  $\sim 0.71\%$  of the total magnetic moment,

(ii) relativistic corrections from the expectation values of positive energy states of the Lorentz transformation of the intrinsic variables in the BS amplitude (the term  $R_+$  in eq. (111)), which is found to be negative, i.e.,  $R_+ = -9.75 \cdot 10^{-4} (e/2m)$ ,

(iii) the term  $\Delta\mu_-$ , and the sum of transitions between states with negative energy ( $u^-, w^-$ ), and transitions between  $P$  states themselves, and a part coming from normalization effects (cf. eq. (110)); this is a positive contribution with  $\Delta\mu_- = 2.99 \cdot 10^{-4} (e/2m)$  to the total moment.

An analysis of our numerical results obtained for the off-diagonal expectation values between the  $S$  and  $P$  partial wave states shows that, in contrast to eqs. (98)-(99), the contributions of terms like pair creation corrections in this case do not compensate each other and give a total contribution to the magnetic moment  $\sim 0.35\%$ , which is almost 50% of the total relativistic correction.

## 5 Concluding remarks

In this paper we have investigated in some detail the numerical solution of the Bethe-Salpeter equation [9] with a realistic one-boson exchange interaction. Special attention has been paid to a study of the relation of the partial BS amplitudes to the nonrelativistic wave functions and to the covariant description of the static characteristics of the deuteron. In our analysis we consider various bases used in defining the partial BS amplitudes and the transition from one basis to another. The representation based on the complete set of the Dirac matrices and their bilinear combinations is found to be extremely convenient in computing the deuteron observables and processes with the deuteron [31] since in this case the dependence on the kinematical variables is mainly included in the definition of the partial amplitudes (except for one spinor propagator, which usually appears when computing diagrams for concrete processes, see ref. [9]), and the matrix structure of the corresponding matrix element is almost

independent of the intrinsic deuteron variables. However, in this representation an analysis of the deuteron structure in terms of familiar  $S$ ,  $D$ , etc. components and an investigation of the correspondence of the obtained results with their nonrelativistic analogues is straitened. For this sake it is more convenient to use the  $\rho$  spin classification of the amplitudes for which a physical interpretation of results is easier. In order to combine the advantages of these two representations the corresponding unitary transformation has been presented explicitly, cf. eq. (51). With this at hand, calculations of various processes can be performed easily in the basis of the Dirac matrixes and the final expression may be treated in terms of the  $\rho$  spin partial amplitudes by utilizing eq. (51). This scheme of calculation has been employed in order to compute the pseudo-probabilities of different partial states and to find the nonrelativistic limit of the amplitudes. In section III different methods of comparison of our amplitudes with the nonrelativistic  $S$  and  $D$  waves are presented. Apparently, the most appropriate way to define the nonrelativistic limits of the BS amplitudes is to use the relation (64), which is based on an analysis of the behavior of the BS vertex functions in dependence on  $p_0$  and  $|\mathbf{p}|$  and on the nonrelativistic relation between the vertices and wave functions in the momentum space. Numerical results, displayed in figs. 8 and 9, show that the generalized BS wave functions (64) are close to the nonrelativistic ones only for moderate values of  $|\mathbf{p}|$ , while a difference occurs for  $|\mathbf{p}| \geq m$ . This means that for rough estimates of possible relativistic effects one may calculate the corresponding nonrelativistic expressions by utilizing the wave functions (64) instead of the nonrelativistic  $S$  and  $D$  waves. Obviously, for a consistent investigation of the relativistic corrections it is necessary to use the covariant calculations with complete BS amplitudes.

We have investigated the quadrupole and magnetic moments of the deuteron within the BS formalism by computing in the Breit frame the matrix elements of the electromagnetic current of the deuteron. In our analysis we considered all the possible relativistic effects connected with both the Lorentz transformation from the rest frame of the deuteron to the Breit frame and with the dependence of the amplitude on the relative energy  $p_0$ . By utilizing results of the investigation of the properties of the BS amplitudes performed in section III and their nonrelativistic limits, the static moments of the deuteron have been presented as a sum of two terms: one of them possesses a direct nonrelativistic analogue, the other one is of pure relativistic nature. We pay special attention to the contribution of the nondiagonal expectation values between  $S$  and  $P$  configurations which are thought to include into the relativistic calculations the effects of pair currents, which are widely discussed in nonrelativistic theories. It has been shown that for the quadrupole moment the different partial transitions between  $S$  and  $P$  components possess a noticeable magnitude, however, their summed contribution is found to be negligibly small (see eqs. (98) and (99)), whereas for the magnetic moment these matrix elements give almost 50% of the relativistic effects. We obtain a good description of the experimental data for the magnetic moment. The computed value of the quadrupole moment is below the experimental data by about 6%. That indicates that even a consistent relativistic computation does not perfectly describe the data in the impulse approximation. Probably, an adjustment of the operator of the

electromagnetic current of the deuteron is needed, e.g., by including additional terms not accounted for within the present approach, such as meson exchange currents with two-meson exchange diagrams or  $\Delta$  isobars [10].

## 6 Summary

In summary an analysis of the properties of the partial Bethe-Salpeter amplitudes, obtained as numerical solution of the BS equation with a realistic interaction, has been performed. In order to compare relativistic amplitudes with the nonrelativistic wave functions a method, based on the comparative analysis of the observables, has been developed. The static characteristics of the deuteron, i.e., the quadrupole and magnetic moments, have been computed within the Bethe-Salpeter formalism with satisfactory accuracy. Our results let us trust in the reliability of our approach, so that it can be used for other tasks, e.g., when tackling the nuclear effects in extracting the neutron structure function from scattering experiments off the deuteron.

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## Appendix I

The matrix form of the spin-angular functions  $\mathcal{V}_{\mathcal{M}}^{\alpha}(\mathbf{p})$ , eq. (47), may be obtained explicitly by replacing the outer product of the free nucleon spinors  $U_{s_i}^{\rho}(\mathbf{p})$  by their direct product,  $U_{s_1}^{\rho_1}(\mathbf{p}) \otimes U_{s_2}^{\rho_2 T}(-\mathbf{p})$ . The BS amplitude takes then the form

$$\chi_D(p_0, \mathbf{p}) U_C = \sum_{\alpha} \phi_{\alpha}(p_0, |\mathbf{p}|) \Gamma_{\mathcal{M}}^{\alpha}(\mathbf{p}) U_C, \quad (113)$$

with  $\Gamma_{\mathcal{M}}^{\alpha}(\mathbf{p})$

$$\Gamma_{\mathcal{M}}^{\alpha}(\mathbf{p}) = i^L \sum_{s_1 s_2 m} (LmSs|J\mathcal{M}) \left( \frac{1}{2} s_1 \frac{1}{2} s_2 |Ss \right) Y_{Lm}(\hat{\mathbf{p}}) U_{s_1}^{\rho_1}(\mathbf{p}) U_{s_2}^{\rho_2 T}(-\mathbf{p}), \quad (114)$$

where  $U_C$  is the charge conjugation matrix,  $U_C = i\gamma_2\gamma_0$ .

One can exploit the  $\rho$  spin dependence and replace  $\Gamma_{\mathcal{M}}^{\alpha}(\mathbf{p}) \equiv \Gamma_M^{\tilde{\alpha}, \rho_1 \rho_2}(\mathbf{p})$ , where

$$\begin{aligned} \Gamma_{\mathcal{M}}^{\tilde{\alpha}, ++}(\mathbf{p}) &= \frac{\hat{k}_1 + m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}(\mathbf{p}, \boldsymbol{\xi}) \frac{\hat{k}_2 - m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}}, \\ \Gamma_{\mathcal{M}}^{\tilde{\alpha}, --}(\mathbf{p}) &= \frac{\hat{k}_2 - m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}} \frac{-1 + \gamma_0}{2} \tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}(\mathbf{p}, \boldsymbol{\xi}) \frac{\hat{k}_1 + m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}}, \\ \Gamma_{\mathcal{M}}^{\tilde{\alpha}, +-}(\mathbf{p}) &= \frac{\hat{k}_1 + m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}(\mathbf{p}, \boldsymbol{\xi}) \frac{\hat{k}_1 + m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}}, \\ \Gamma_{\mathcal{M}}^{\tilde{\alpha}, -+}(\mathbf{p}) &= \frac{\hat{k}_2 - m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}} \frac{1 - \gamma_0}{2} \tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}(\mathbf{p}, \boldsymbol{\xi}) \frac{\hat{k}_2 - m}{\sqrt{2E_{\mathbf{p}}(m + E_{\mathbf{p}})}}, \end{aligned} \quad (115)$$

with  $\tilde{\alpha} \in \{L, S, J\}$ .

The spin-angular structures for some partial waves are shown in Table 3. Here  $\boldsymbol{\xi}_{\mathcal{M}}$  is the polarization vector of the deuteron with the components in the rest frame given by

$$\boldsymbol{\xi}_{+1} = (-1, -i, 0)/\sqrt{2}, \quad \boldsymbol{\xi}_{-1} = (1, -i, 0)/\sqrt{2}, \quad \boldsymbol{\xi}_0 = (0, 0, 1), \quad (116)$$

and the four-vector  $\xi_{\mathcal{M}} = (0, \boldsymbol{\xi}_{\mathcal{M}})$ .

## Appendix II

Below we list the explicit form of the quantities  $C^{a,b}$  and  $H_i^{\alpha',\alpha}$  in eq. (101). By introducing new functions  $G_i^{a,b}$  the mentioned quantities are expressed as follows:

$$\begin{aligned}
 \tau = u^\pm, v_s^e : \quad C^\tau &= \frac{\sqrt{6} m}{12 M} \left[ G_1^\tau + 4G_2^\tau - 4G_3^\tau - G_4^\tau - 4G_5^\tau \right], \\
 \tau = u^\pm, v_s^o : \quad C^\tau &= \frac{\sqrt{6} m}{15 M} \left[ -G_6^\tau \mp G_7^\tau \mp G_8^\tau \pm G_9^\tau - G_{27}^\tau \right], \\
 \tau = w^\pm, v_s^e : \quad C^\tau &= \frac{\sqrt{3} m}{12 M} \left[ G_{10}^\tau + G_{11}^\tau + 4G_3^\tau + G_4^\tau + 4G_5^\tau \right], \\
 \tau = w^\pm, v_s^o : \quad C^\tau &= \frac{\sqrt{3} m}{15 M} \left[ G_{12}^\tau \pm G_{13}^\tau \pm G_{14}^\tau \mp G_{15}^\tau + G_{28}^\tau \right], \\
 \tau = u^\pm, v_t^e : \quad C^\tau &= \frac{\sqrt{3} m}{15 M} \left[ \pm G_{16}^\tau - G_{17}^\tau + G_8^\tau - G_9^\tau \pm G_{27}^\tau \right], \\
 \tau = u^\pm, v_t^o : \quad C^\tau &= \mp \frac{\sqrt{3} m}{3 M} \left[ G_{20}^\tau + G_{21}^\tau + G_3^\tau + \frac{1}{4}G_4^\tau + G_5^\tau \right] \mp \frac{\sqrt{3}}{3} \kappa G_{22}^\tau, \\
 \tau = w^\pm, v_t^e : \quad C^\tau &= \frac{\sqrt{6} m}{15 M} \left[ \pm G_{23}^\tau - G_{24}^\tau + G_{18}^\tau - G_{19}^\tau \pm G_{29}^\tau \right], \\
 \tau = w^\pm, v_t^o : \quad C^\tau &= \mp \frac{\sqrt{6} m}{3 M} \left[ G_{25}^\tau + G_{26}^\tau + G_3^\tau + \frac{1}{4}G_4^\tau + G_5^\tau \right] \pm \frac{\sqrt{6}}{6} \kappa G_{22}^\tau,
 \end{aligned}$$

where the  $G_i^{\alpha,\alpha'}$  are integrals of the form

$$N \int dp_4 d|\mathbf{p}| |\mathbf{p}|^2 A_i(p_4, |\mathbf{p}|) [B_i Y_\alpha(p_4, |\mathbf{p}|)] Y_{\alpha'}(p_4, |\mathbf{p}|),$$

and  $A_i(p_4, |\mathbf{p}|)$  are scalar functions;  $B_i$  may be either a differential operator of the type  $\partial/\partial p_4$ ,  $\partial/\partial |\mathbf{p}|$  or a scalar function ( $p_4 = -ip_0$ ), which are summarized in the following tabular form:

$G_i$	$A_i(p_4,  \mathbf{p} )$	$B_i$
1	$(E - m)Mm/( \mathbf{p} E^2)$	$-\omega_\alpha$
2	$-(E - m)mp_4^2/( \mathbf{p} E^2)$	1
3	$ \mathbf{p} p_4$	$\partial/\partial p_4$
4	$M$	$-\omega_\alpha \partial/\partial p_4$
5	$-p_4^2$	$\partial/\partial  \mathbf{p} $
6	$(E - m)(2E + 3m)p_4/( \mathbf{p} E)$	1
7	$(E - m)(E^2 + 2mE + 2m^2)Mp_4/( \mathbf{p} E^3)$	1
8	$ \mathbf{p} (3E + 2m)/(2E)$	$-\omega_\alpha \partial/\partial p_4$
9	$(3E + 2m)Mp_4/E$	$\partial/\partial  \mathbf{p} $
10	$(2E + m)mM/( \mathbf{p} E^2)$	$-\omega_\alpha$
11	$-(2E + m)mp_4^2/( \mathbf{p} E^2)$	1
12	$(2E^2 - 2mE - 3m^2)p_4/( \mathbf{p} E)$	1
13	$(E^3 - 2mE^2 + 4m^3)Mp_4/( \mathbf{p} E^3)$	1
14	$(3E - 4m) \mathbf{p} /(2E)$	$-\omega_\alpha \partial/\partial p_4$
15	$(3E - 4m)Mp_4/E$	1
16	$(E - m)(7E + 3m)p_4/( \mathbf{p} E)$	1
17	$2(E - m)M(2E^2 - mE - m^2)p_4/( \mathbf{p} E^3)$	1
18	$ \mathbf{p} (3E - m)/(2E)$	$-\omega_\alpha \partial/\partial p_4$
19	$(3E - m)p_4M/E$	$\partial/\partial  \mathbf{p} $
20	$(E - m)^2M/(4 \mathbf{p} E^2)$	$-\omega_\alpha$
21	$-(E - m)^2p_4^2/( \mathbf{p} E^2)$	1
22	$ \mathbf{p} /(2E)$	$-\omega_\alpha$
23	$(7E^2 + 2mE - 3m^2)p_4/( \mathbf{p} E)$	1
24	$(4E^3 + 3mE^2 - m^3)Mp_4/( \mathbf{p} E^3)$	1
25	$(E^2 + mE + m^2)M/(4 \mathbf{p} E^2)$	$-\omega_\alpha$
26	$-(E^2 + mE + m^2)p_4^2/( \mathbf{p} E^2)$	1
27	$(3E + 2m)p_4$	$\partial/\partial  \mathbf{p} $
28	$(3E - 4m)p_4$	$\partial/\partial  \mathbf{p} $
29	$(3E - m)p_4$	$\partial/\partial  \mathbf{p} $

Analogously, the functions  $H_i^{\alpha, \alpha'}$  ( $H_i^{\alpha, \alpha} \equiv H_i^\alpha$ ) are of the same structure as  $G_i^{\alpha, \alpha'}$  with

$i$	$A_i(p_4,  \mathbf{p} )$	$B_i$	$i$	$A_i(p_4,  \mathbf{p} )$	$B_i$
1	$\frac{1}{2}(1 - m/E)$	$\omega_\alpha$	6	$Mm^2/E^2$	1
2	$\frac{1}{2}(1 - M/2E)$	$\omega_\alpha$	7	$-p_4^2m/E^2$	1
3	$-p_4/E$	1	8	$p_4m^3/E^3$	1
4	$p_4m/E$	1	9	$ \mathbf{p} ^2m/E$	$\partial/\partial p_4$
5	1/2	$\omega_\alpha$	10	$ \mathbf{p} p_4m/E$	$\partial/\partial  \mathbf{p} $



## References

- [1] J. Flescher and J. Tjon, Nucl. Phys. **B 84**, 375 (1975); Phys. Rev. **D 15**, 2537 (1977);  
M.J. Zuilhof and J.A. Tjon, Phys. Rev. **C 22**, 2369 (1980);  
E. van Faassen and J. Tjon, Phys. Rev. **C 33**, 2105 (1986).
- [2] F. Gross, Phys. Rev. **186**, 1448 (1969);  
W.W. Buck and F. Gross, Phys. Rev. **D 20**, 2361 (1979).
- [3] F. Gross, J.W. Van Orden and K. Holinde, Phys. Rev. **C 45**, 2094 (1992).
- [4] P.L. Chung, F. Coester, B.D. Keister and W.N. Polyzou, Phys. Rev. **C 37**, 2000 (1988).
- [5] J. Carbonell and V.A. Karmanov, Nucl. Phys. **A 581**, 625 (1995).
- [6] F. Gross, Talk given at the 9-th Amsterdam miniconference "Electromagnetic studies of the deuteron", NIKHEF, Amsterdam, February 1-2, 1996.
- [7] M. Lacombe et al., Phys. Rev. **C 21**, 861 (1980).
- [8] R. Machleid, K. Holinde and Ch. Elster, Phys. Rep. **149**, 1 (1987).
- [9] A.Yu. Umnikov and F.C. Khanna, Phys. Rev. **C 49**, 2311 (1994);  
A.Yu. Umnikov, L.P. Kaptari, K.Yu. Kazakov and F.C. Khanna, Phys. Lett. **B 334**, 163 (1994).
- [10] E.L. Lomon, Ann. Phys. **125**, 309 (1980).
- [11] K. Tamura, T. Niwa, T. Sato and H. Ohtsubo, Nucl. Phys. **A 536**, 597 (1992).
- [12] M. Gari, H. Hyuga and B. Sommer, Phys. Rev. **C 14**, 2196 (1976);  
E. Hadjimichael, Nucl. Phys. **A 312**, 341 (1978).
- [13] T. Sato, M. Kobayashi and H. Ohtsubo, Prog. Theor. Phys. **68**, 840 (1982);  
V.V. Burov and V.N. Dostovalov, Z. Phys. **C 326**, 245 (1987);  
M. Chemtob, in: Mesons in Nuclei, Amsterdam, North-Holland (1976), p. 555;  
M. Gari and H. Hyuga, Z. Phys. **A 277**, 291 (1976);  
M. Kirchbach, E. Truhlik, Particles and Nuclei **17**, 224 (1980);  
M. Chemtob and M. Rho, Nucl Phys. **A 163**, 1 (1971).
- [14] M.J. Zuilhof and J.A. Tjon, Phys. Rev. **C 22**, 2369 (1980).
- [15] G. Rupp and J.A. Tjon, Phys. Rev. **C 41**, 472 (1990).
- [16] N. Honzawa and S. Ishida, Phys. Rev. **C 45**, 47 (1992).
- [17] F. Schlumpf and S. Brodsky, SLAC-PUB-95-6860; UMDP-PP-95122 (e-print archives hep-ph/9506276).

- [18] B. Desplanques, V.A. Karmanov and J.-F. Mathiot, Nucl. Phys. **A 589**, 697 (1995).
- [19] S.G. Bondarenko, V.V. Burov, M. Beyer and S.M. Dorkin, *to be published*.
- [20] S.G. Bondarenko et al., in: Proc. of the III Int. Symp. "Dubna, Deuteron 95", Dubna (1995), *in press*.
- [21] S.E. Kuhn et al., "The Polarized Structure Function  $G_{1n}$  and the  $Q^2$  Dependence of the Gerasimov-Drell-Hearn Sum Rule for the Neutron", CEBAF Proposal No. 93 - 009;  
J.P. Chen, S. Gilad, Zh. Li and C.S. Whichat (spokesmen), "Helicity Structure of Pion Photoproduction on Polarized Deuteron and the GDH Sum Rule for the Neutron", CEBAF Proposal, 94-117.
- [22] S.B. Gerasimov, Yad. Fiz. **2**, 598 (1965); *ibid.* **5**, 1263 (1967);  
S.D. Drell and A.C. Hearn, Phys. Rev. Lett. **16**, 908 (1966).
- [23] M. Gourdin, Nuovo Cimento **28**, 533 (1963).
- [24] T.W. Donnelly and A.S. Raskin, Ann. Phys. **169**, 247 (1986).
- [25] S.I. Nagorny et al., Sov. J. Nucl. Phys. **49**, 465 (1989).
- [26] J.J. Kubis, Phys. Rev. **D 6**, 547 (1972).
- [27] H.F.K. Zingl, L. Mathelitsch and M.I. Haftel, Acta Phys. Austr. **53**, 29 (1981);  
L. Mathelitsch and H.F.K. Zingl, Nuovo Cimento **44 A**, 81 (1978).
- [28] J. Hockert et al., Nucl. Phys. **A 217**, 14 (1973).
- [29] J.A. Lock and L.L. Foldy, Ann. Phys. **93**, 276 (1975).
- [30] R.F. Code and N.F. Rammsey, Phys. Rev. **A 4**, 1954 (1971);  
R.V. Reid and M.L. Vaida, Phys. Rev. Lett. **29**, 494 (1972); **34**, 1064E (1975); Phys. Rev. **A 7**, 1841 (1973).
- [31] A.Yu. Umnikov, L.P. Kaptari, and F. Khanna, Phys. Lett. **B 334**, 163 (1994);  
L.P. Kaptari, A.Yu. Umnikov, B. Kämpfer and F. Khanna, Phys. Lett. **B 351**, 400 (1995);  
F.C. Khanna, L.P. Kaptari and A. Yu. Umnikov, Czech. J. Phys. **45**, 363 (1995);  
A.Yu. Umnikov, L.P. Kaptari, K.Yu. Kazakov, F.C. Khanna, Preprint Alberta Thy-29-94 (e-print archives hep-ph/9410241).

Table 1: *The deuteron partial amplitudes and their transformation properties.*

$^{2S+1}L_1$	$P_1$	$A_1^0$	$V_1$	$A_0$	$A_2$	$T_1^0$	$T_0$	$T_2$
$L$	1	1	1	0	2	1	0	2
$S$	0	0	1	1	1	1	1	1
$\mathcal{K}$	+	+	+	+	+	-	+	+
$\Pi$	+	-	+	+	+	-	+	+

Table 2: *The pseudo-probabilities of the partial waves in the deuteron.*

wave	$u^+$	$w^+$	$u^-$	$w^-$
$P_\alpha(\%)$	95.014	5.106	-0.002	-0.003
wave	$v_s^e$	$v_t^o$	$v_s^o$	$v_t^e$
$P_\alpha(\%)$	-0.010	-0.082	-0.015	-0.008

Table 3: *Spin-angular functions  $\tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}$  for the deuteron channel.*

$\tilde{\alpha}$	$\sqrt{8\pi} \tilde{\Gamma}_{\mathcal{M}}^{\tilde{\alpha}}$
$^3S_1$	$\hat{\xi}_{\mathcal{M}}$
$^3D_1$	$-\frac{1}{\sqrt{2}} \left[ \hat{\xi}_{\mathcal{M}} + \frac{3}{2}(\hat{k}_1 - \hat{k}_2)(p\xi_{\mathcal{M}}) \mathbf{p} ^{-2} \right]$
$^3P_1$	$\sqrt{\frac{3}{2}} \left[ \frac{1}{2}\hat{\xi}_{\mathcal{M}}(\hat{k}_1 - \hat{k}_2) - (p\xi_{\mathcal{M}}) \right]  \mathbf{p} ^{-1}$
$^1P_1$	$\sqrt{3}(p\xi_{\mathcal{M}}) \mathbf{p} ^{-1}$

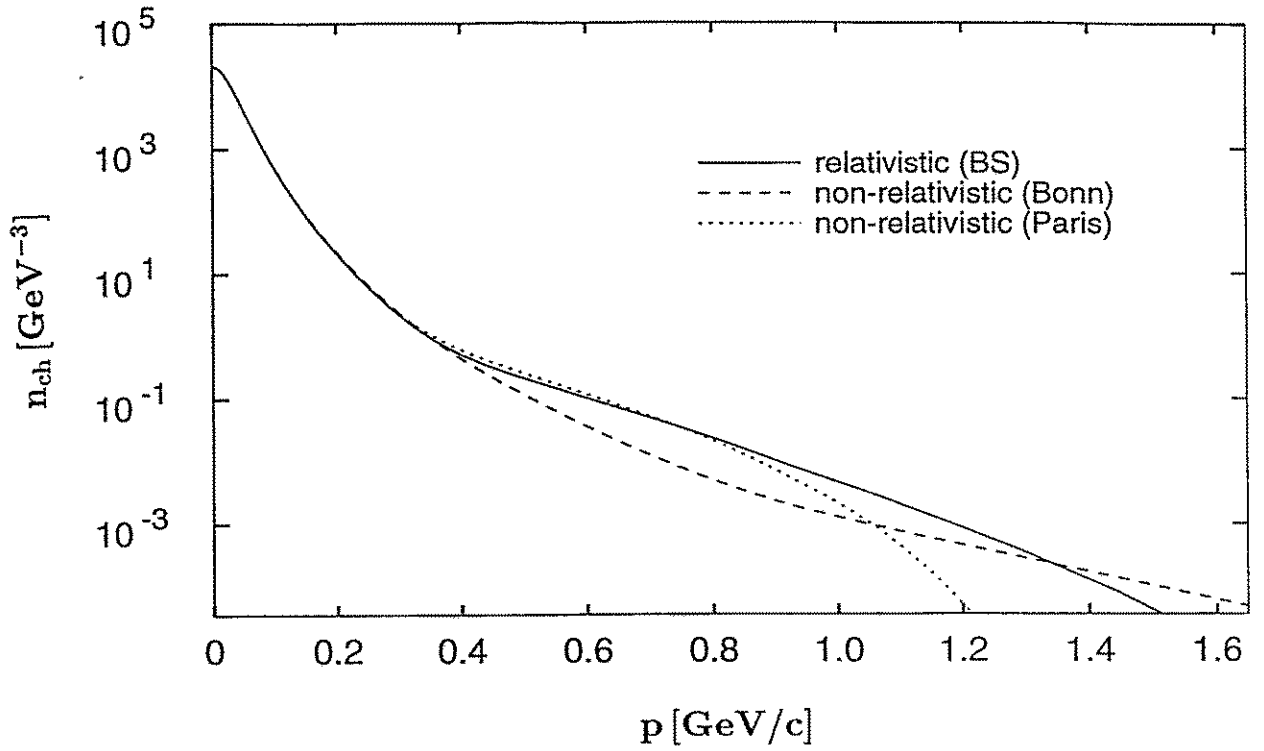


Figure 1: The nucleon density in the deuteron computed within the BS formalism in comparison with the nonrelativistic results.

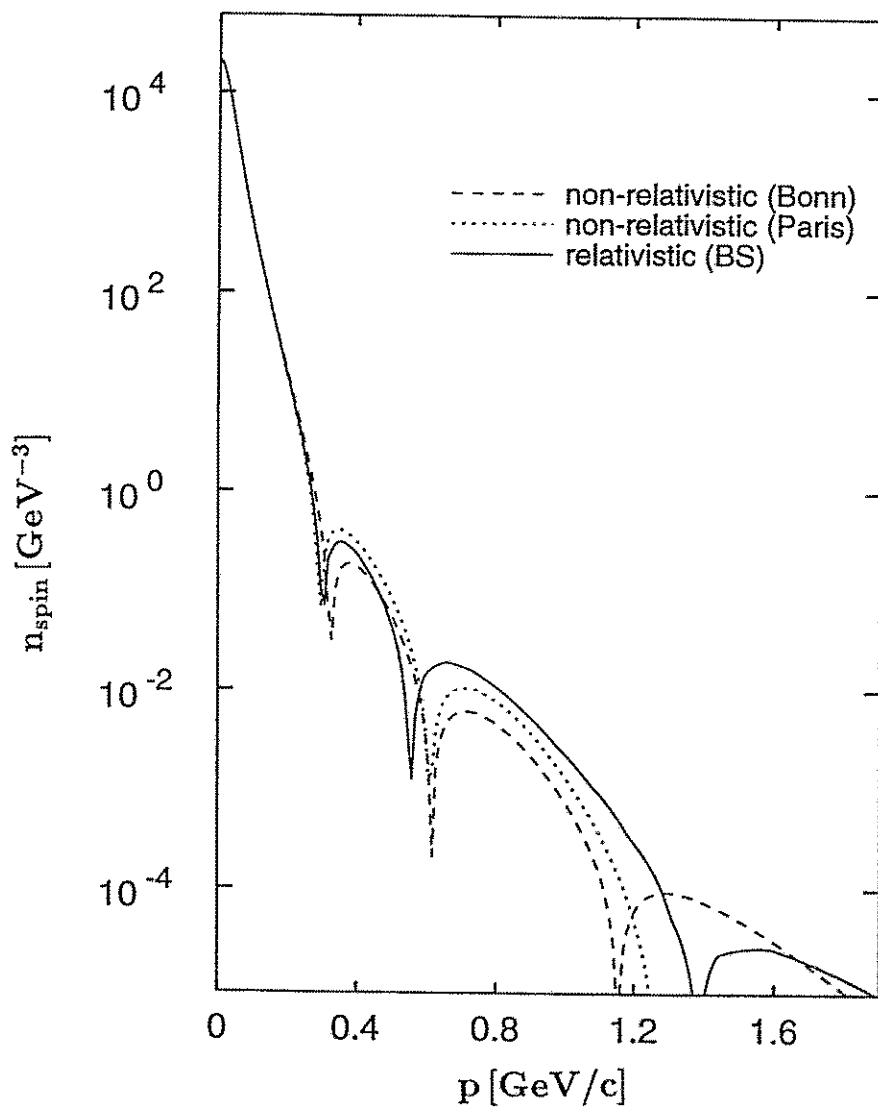


Figure 2: The nucleon spin distribution in the deuteron computed within the BS formalism in comparison with the nonrelativistic results.

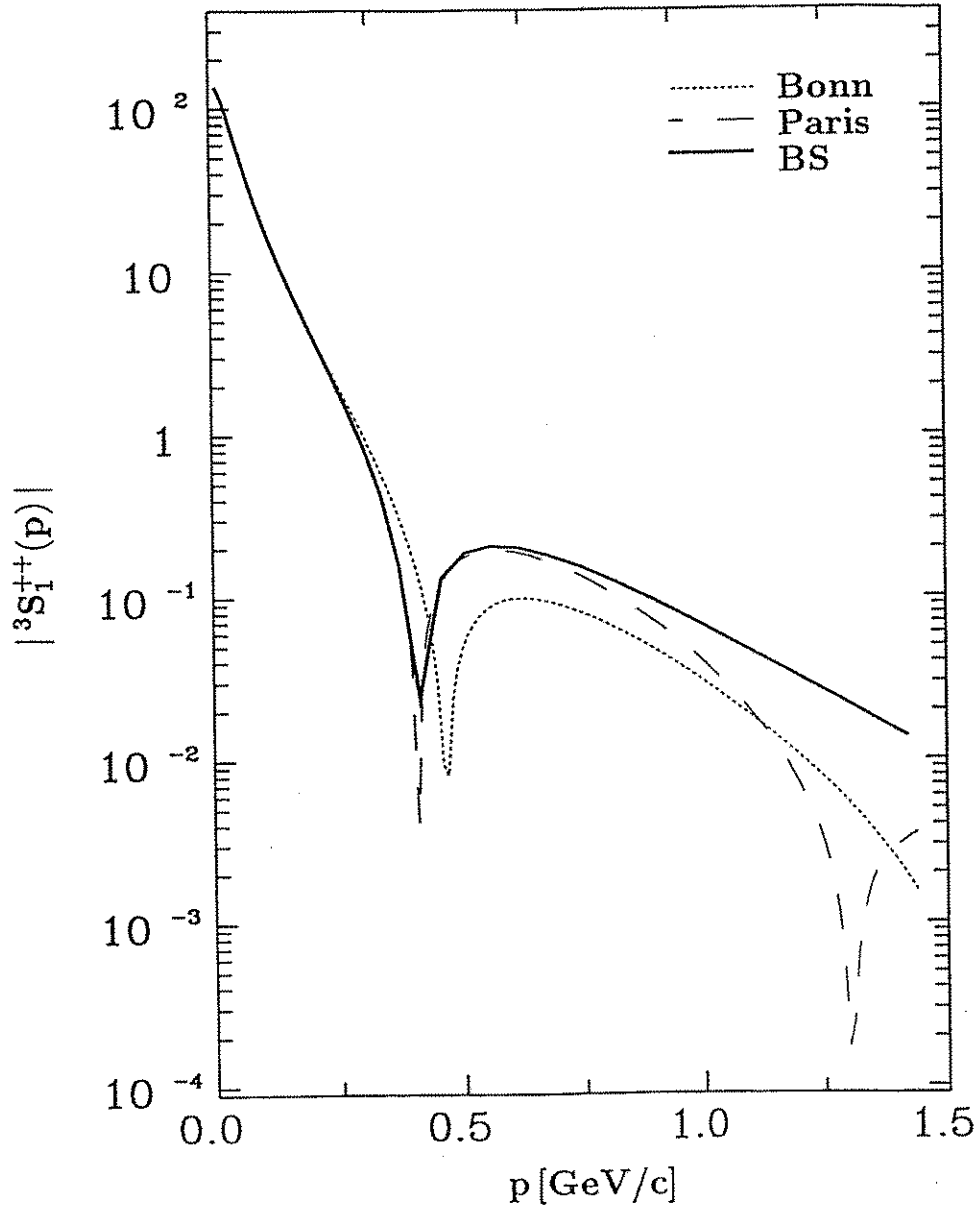


Figure 3: The momentum dependence of the  ${}^3S_1^{++}$  component defined by eq. (55) (solid line) in comparison with the corresponding nonrelativistic wave functions with Bonn and Paris potentials (dotted and dashed lines, respectively).

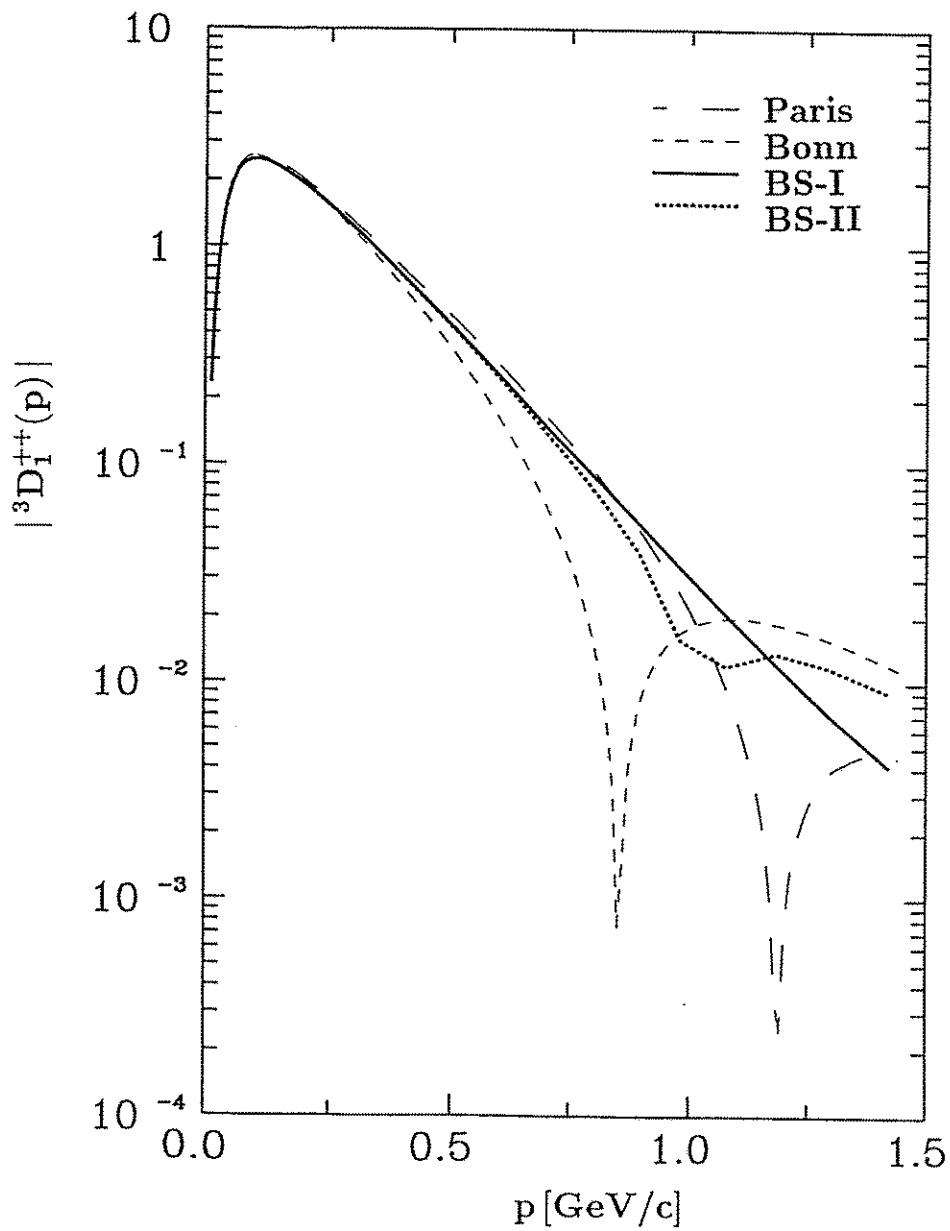


Figure 4: The momentum dependence of the  ${}^3D_1^{++}$  component. The solid line (BS-I) depicts the result of computation by (55); the dotted line (BS-II) includes the contribution of  $P$  waves (see text); short-dashed and long-dashed lines depict the nonrelativistic wave functions with Bonn and Paris potentials, respectively.

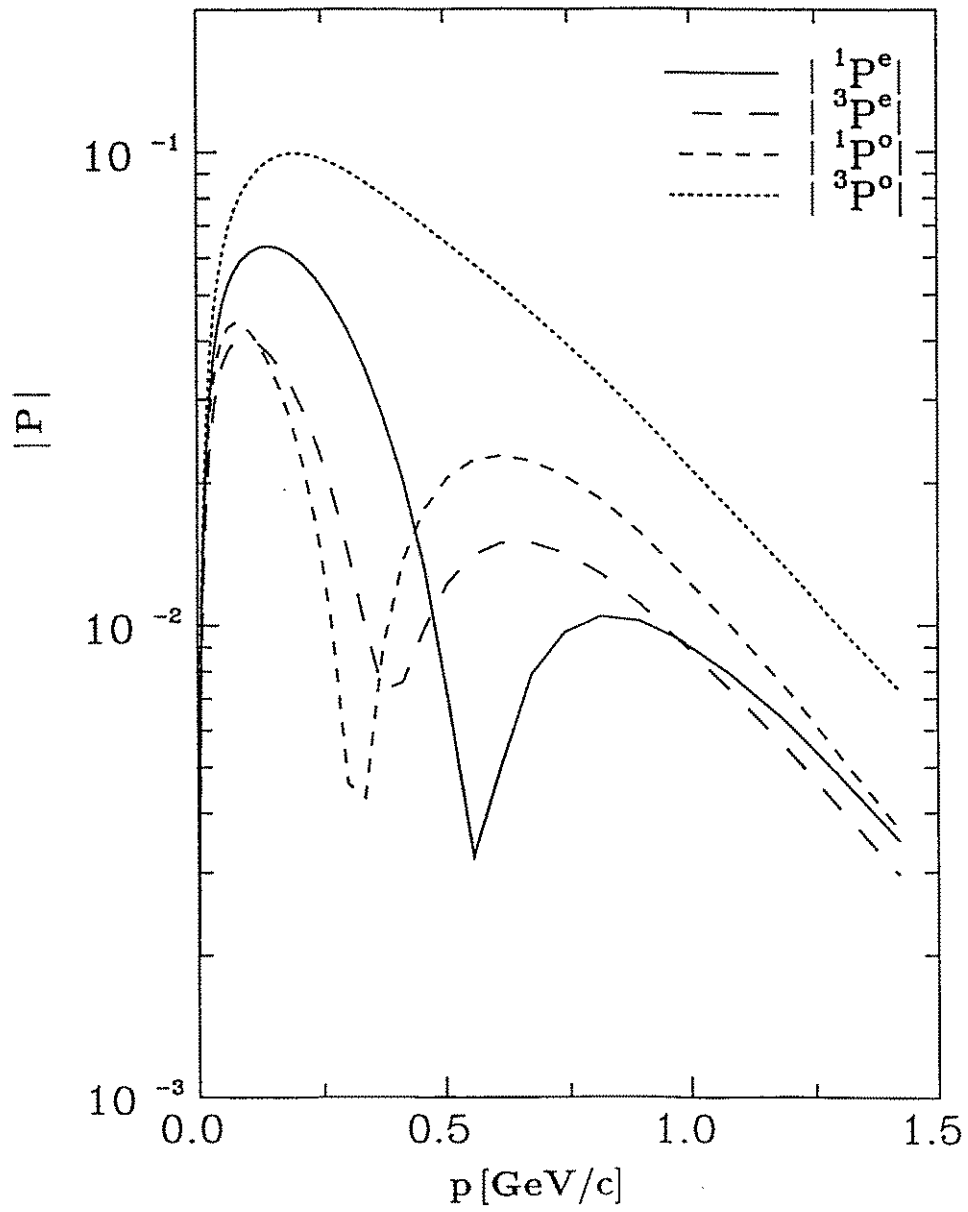


Figure 5: The momentum dependence of the  $P$  waves defined by eq. (55).



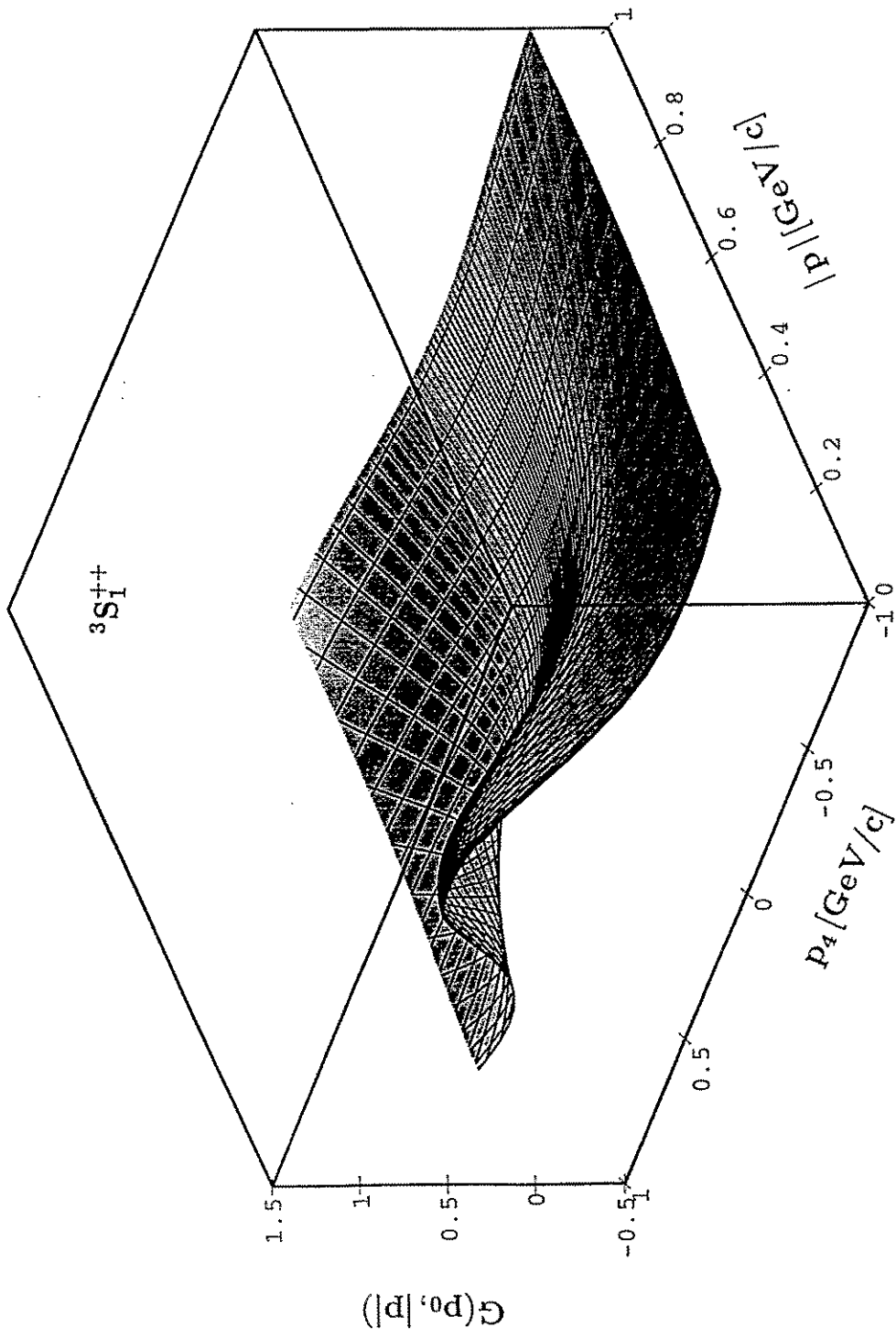


Figure 6: The behavior of the vertex function  $G(p_0, |p|)$  for the  ${}^3S_1^{++}$  configuration in the deuteron in dependence on  $p_4$  and  $|p|$ .

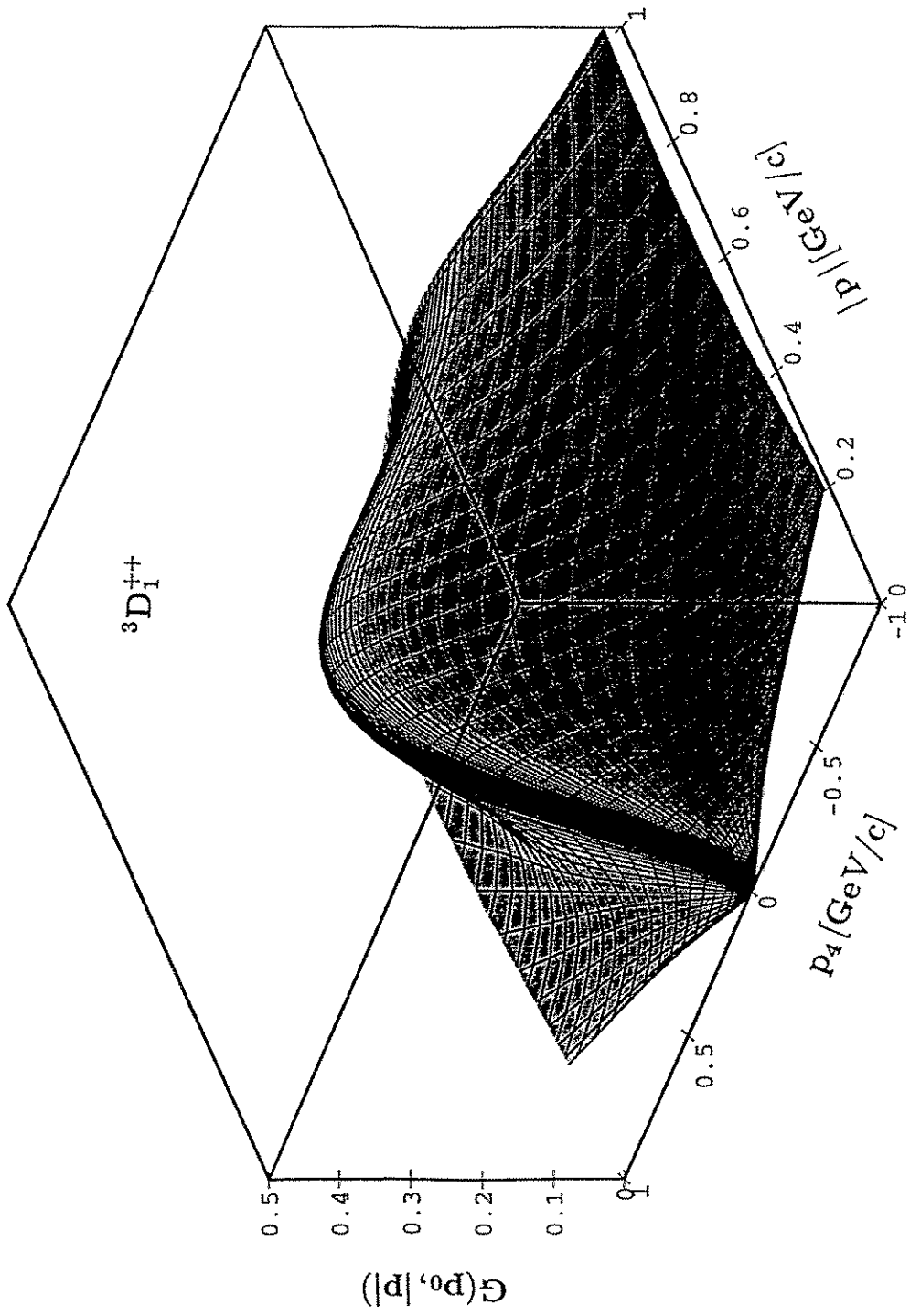


Figure 7: The same as fig. 6 but for the  ${}^3D_1^{++}$  configuration.

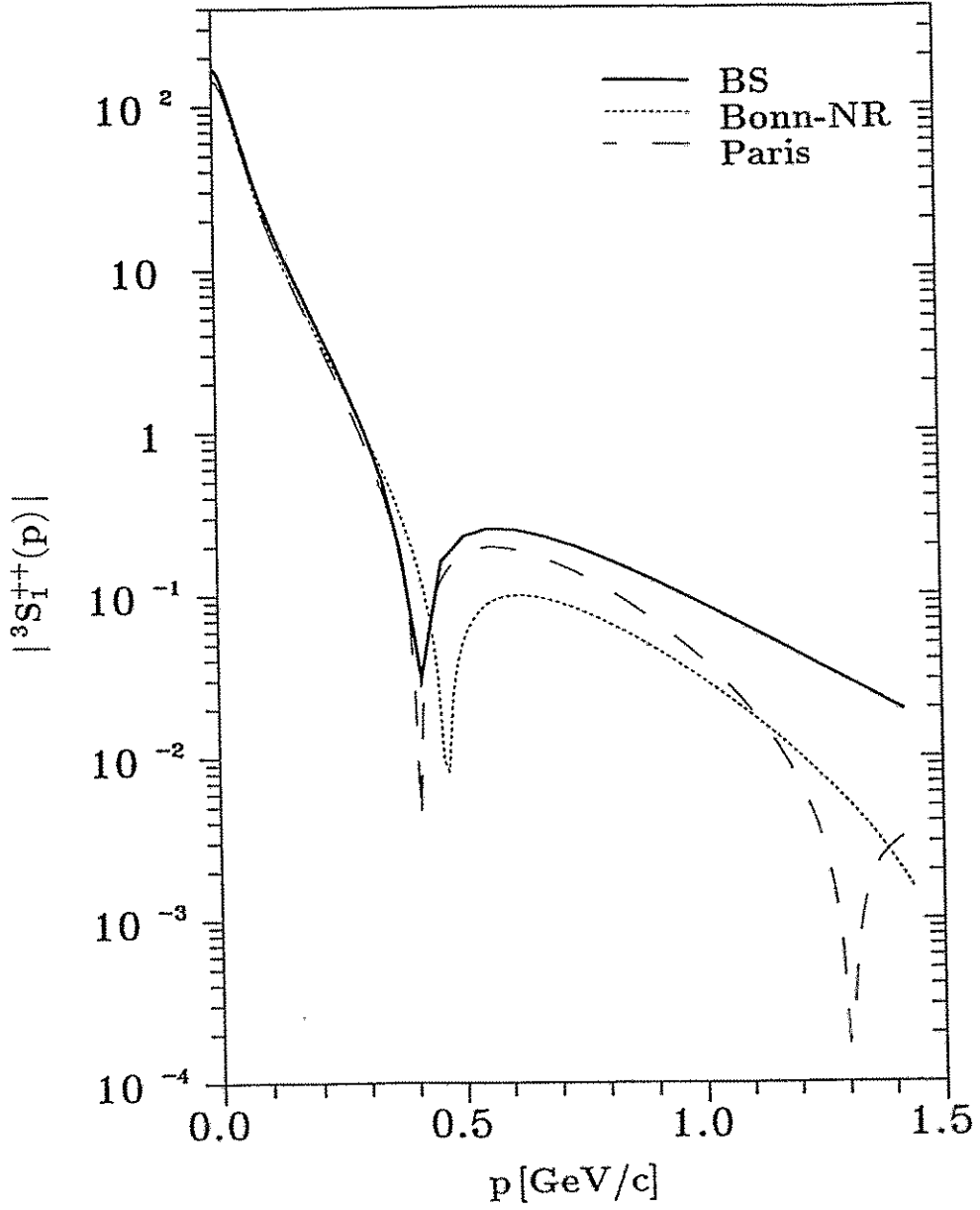


Figure 8: The nonrelativistic limit of the  ${}^3S_{1^{++}}$  component defined by eq. (64) (solid line) in comparison with the nonrelativistic wave functions with Bonn and Paris potentials (dotted and dashed lines, respectively).

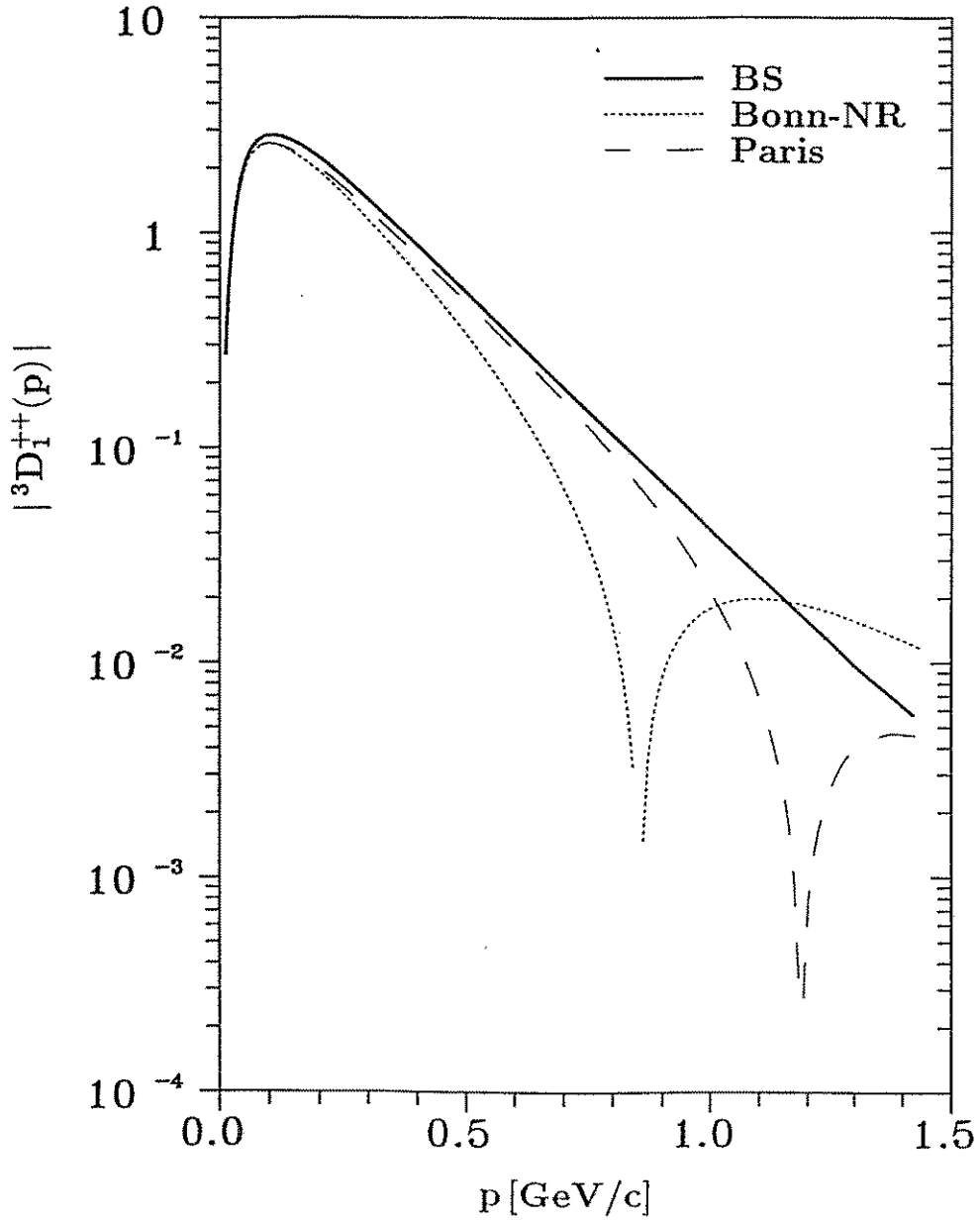


Figure 9: The same as fig. 8 but for the  ${}^3D_1^{++}$  components.