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The $p-$Center Problem under Locational Uncertainty of Demand Points

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Abstract

The p-center problem is finding the location of p facilities among a set of n demand points such that the maximum distance between any demand point and its nearest facility is minimized. In this paper, we study this problem in the context of uncertainty, that is, the location of the demand points may change in a region like a disk or a segment, or belong to a finite set of points. We introduce *Max-p-center* and *Min-p-center* problems which are the worst and the best possible solutions for the p-center problem under such locational uncertainty. We propose approximation and parameterized algorithms to solve these problems under the Euclidean metric. Further, we study the MinMax Regret 1-center problem under uncertainty and propose a linear-time algorithm to solve it under the Manhattan metric as well as an $O(n^4)$ time algorithm under the Euclidean metric.

Keywords: Facility location, p-center, Uncertainty, Regret, Robustness, Approximation algorithms.

1. Introduction

The *p-center* problem (also called *k-center* problem) is a classic facility location problem with many real-world applications, such as locating emergency

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facilities. In this problem, given a set of n demand points (customers) and a number $p < n$, the goal is to place p facilities (centers) such that the maximum distance between any demand point and its nearest center is minimized. Most of the studies on this problem have an important assumption that the location of the demand points is precise. So, the cost between the centers and customers is certain and exact. However, there are considerable sources of uncertainty and/or error in the real world, such as modeling the problems, data gathering, computations and implementing outcomes of the algorithm [1, 2, 3]. For example, when traveling time is considered as the cost between the centers and customers, it may vary because of traffic jams, weather conditions, etc. Also, it is possible the location of the customers is uncertain –called *locational uncertainty* [4]. For example, consider a set of mobile devices such as cell phones or laptops that move in predefined rooms, and receive signals (like Wifi) from some access points. In this paper, we focus on this challenge and investigate the p-center problem under uncertainty.

The p-center problem with the precise location of demand points is an NP-hard problem for both the Euclidean and Manhattan metrics [5, 6]. So, there is no polynomial-time algorithm to solve it in the general case, and only limited versions of the problem have been solved in polynomial time, such as small and constant numbers of p. For $p = 1$, the problem is called the *smallest enclosing circle*, and efficient (almost linear time) algorithms have been proposed for solving it [7, 8, 9]. For $p = 2$, the problem was solved in $O(n \log^2 n)$ time using a divide and conquer approach [10]. The rectilinear 3-center problem was solved optimally in linear time [11]. Different variations of the problem such as the p-center problem on trees [12, 13] and on a line [14] have been studied as well. Further, for the general cases of the p-center problem, approximation and heuristic approaches have been proposed [15]. All of these studies have considered the locations of the customers to be exact, and so assume a certain distance function between the centers and demand points.

In this paper, we assume the location of the demand points is uncertain and may change in a given region like a disk or a segment, or belongs to a finite set of potential candidates. First, we consider two natural extensions of the p-center problem for uncertain demand points, called Max-p-center and Min-p-center. Max-p-center is the p-center for the worst replacement of the demand points in their corresponding regions, and Min-p-center is the same for the best replacement. After reviewing related studies in the next section, we define Max-p-center and Min-p-center problems formally in Section 3. We present a simple $2-$approximation algorithm to solve Max-p-center problem under the Euclidean metric when the regions are disjoint disks or a set of discrete points. Also, we present a $(1 + \frac{2}{k+2})-$approximation algorithm when the regions are *k-separable* (See Definition 1). Further, we consider the Min-p-center problem under the Euclidean metric and present a $(1 + \frac{2}{k})-$approximation algorithm when the regions of uncertainty are k-separable disks or a set of discrete points. In Section 4, we introduce a new extension of the p-center problem under uncertain demand points, called *MinMax Regret*. The regret is the difference between the cost (maximum distance) of a given solution and the cost (maximum distance) of the optimal solution for a particular placement of the uncertain points. The worst case of the regret between all possible placements of the uncertain parameters is called *MaxRegret*. We study this problem only for the case $p = 1$, and present a linear-time algorithm to solve it under the Manhattan metric when the regions of uncertainty are horizontal segments. Also, we present an $O(n^4)$ time algorithm for solving the MinMax Regret 1-center problem under the Euclidean metric when the regions of uncertainty are n horizontal segments. Finally, in Section 5, we make concluding remarks and discuss future directions.

2. Related work

In addition to the remarkable history of the classic p-center problem and its different variations for certain demand points, some studies have considered the problem under uncertainty. Foul [16] studied the Euclidean 1-center problem under uncertainty for a set of n demand points that have a uniform distribution inside rectangles. Chen et al. [17] studied one dimensional p-

center problem in which the location of each demand point is modeled using m possible locations with a probability distribution function. They proposed an $O(mn \log mn + n \log p \log n)$ time algorithm to solve this problem. Other variations of the p-center problem have been studied under this model of uncertainty. The one dimension 1-center problem, the 1-center problem on the trees and the rectilinear 1-center problem in the plane were studied under this model and several probabilistic algorithms were proposed to solve them [18, 19].

Löffler and van Kreveld [20] presented efficient algorithms for 1-center problem when the uncertainty regions are modeled by squares or disks. The goal is finding a point from each region such that the *Smallest Enclosing Circle* (SEC) of them is minimized or maximized. For a set of regions as the input, the goal is to place a point in each region such that the SEC is minimized or maximized. They proved that when the uncertainty is modeled by disk-shaped regions, the smallest possible SEC and the largest possible SEC can be solved in linear time.

Kouvelis et al. [21] presented an $O(n^4)$ algorithm for the MinMax regret 1-median problem on a tree with n nodes, where the uncertainty is considered as interval numbers. A case of the MinMax regret 1-median on a tree was studied in which the weight of the vertices and the length of the edges are uncertain. Bhattacharya et al. [22] presented a linear time algorithm for this case of the MinMax regret 1-median problem. Also, they presented an $O(n \log^2 n)$ time algorithm to consider the negative weights [23].

Averbakh [24] proved that the MinMax regret 1-median problem on a network with uncertain edge length is strongly NP-hard. Yu et al. [25] studied the problem on general graphs and for a graph with n vertices and m edges presented an $O(mn^2 + n^3 \log n)$ time algorithm when the weight of the vertices is uncertain. They also presented an $O(n \log^2 n)$ time algorithm for the MinMax regret 1-center problem on a tree with n weighted vertices when the weights are uncertain.

Burkard and Dollani [26] presented an algorithm with $O(n^3 \log n)$ time complexity for the MinMax regret 1-center problem when both the length of edges and the weight of vertices are uncertain. They proposed an $O(n \log n)$ time algorithm for the MinMax regret 1-center problem when the length of edges is uncertain. Yu et al. [25] presented an algorithm with $O(mn \log n)$ time complexity for the special case of the MinMax regret 1-center when the weight of the vertices is uncertain. Alipour and Jafari [27] have focused on the expected maximum distance in p-center problem. They introduced the *assigned* and *unassigned uncertain p-center* problems and proposed approximation algorithms for solving them.

Averbakh et al. [28] studied 1-median and weighted 1-center problems in the plane. They presented an $O(n^2 \log^2 n)$ time algorithm for the 1-median problem and an $O(n \log n)$ time algorithm for the 1-center problem under the Manhattan metric. They also studied the weighted 1-center problem in the plane where the weight of demand points is uncertain. They presented an $(n^2 2^{\alpha(n)} \log^2 n)$ time algorithm to solve this problem, where $\alpha(.)$ is the inverse of the Ackermann function.

3. Max- and Min- *p*-center Problems

The p-center problem under locational uncertainty of the demand points is formally defined as follows. Let $\Re = \{R_1, R_2, ..., R_n\}$ be a set of n uncertain demand points, that is, the location of i-th demand point may change in a region R_i. In this paper, we shall refer to them as "regions of uncertainty", and consider three shapes, disk-shaped region, segment-shaped region and discrete sets. Let $I = \{p_1, p_2, ..., p_n\}$ be a *placement* (or say *instance*) of the demand points, i.e., $p_i \in R_i$, for $i = 1, 2, ..., n$. Let $p - center(I)$ denote the optimal solution of the p-center problem for an instance I. Let $C = \{c_1, c_2, ..., c_p\}$, where $c_i \in \mathbb{R}^2$, for $i = 1, 2, \dots, p$, be a set of p centers in the plane. So, the p-center problem is

$$p - center(I) = \min_{C} \max_{p_i \in I} dis(p_i, C),$$

where $dis(p_i, C)$ is the Euclidean distance between p_i and the nearest center in C, i.e., $dis(p_i, C) = \min_{1 \leq j \leq p} dis(p_i, c_j)$. Therefore, *Max-p-center* problem and similarly the Min-p-$center$ problem are the problems of finding the extreme instances I^{max} and I^{min} such that

$$I^{max} : \max_{I} \quad p - center(I),$$
$$I^{min} : \min_{I} \quad p - center(I).$$

Indeed, I^{max} is the worst placement of the demand points which results in the maximum possible for the optimal solution of the p-problem, so it is the *pessimistic* scenario for the arrangement of the demand points. On the other hand, I^{min} denotes the *optimistic* scenario for the demand points and it is the best placement that results in the minimum possible solution for the p-center problem. Therefore, I^{max} and I^{min} together provide a range for the solution value of the p-center problem, and they can help a decision-maker who designs layouts and locates the centers among the demand points under such locational uncertainty.

The decision version of Max-p-center problem can be described as follows. For a given threshold τ, and a set of uncertainty regions $\Re$, whether p–$center(I) \geq \tau$ or not?. Similarly, the decision version of Min-p-center problem asks whether p–$center(I) \leq \tau$?.

Theorem 1. *If $P \neq NP$, the decision version of Max- p-center problem (or Min- p-center problem) for a given set of uncertainty regions, like disk or segment shaped regions or discrete sets, does not belong to NP-complete problems.*

Proof. It is well-known the decision version of the classic p-center problem is NP-complete. Now, by contradiction, if the decision version of Max- p-center (or Min- p-center) problem is an NP-complete problem, we should be able to verify an instance of the problem in polynomial time. However, verifying such an instance is equivalent to determining whether the answer of the p-center problem for such instance is less than or equal to a decision parameter, which results in solving the decision version of the p-center problem in polynomial time, which is a contradiction. Therefore, the decision version of Max- p-center (or Min- p-center) problem belongs to NP-complete problems, i.e., it may belong to the class of NP-hard problems which are not NP-complete or even belong to the class of problems which are not NP-hard. □

Theorem 1 shows the impossibility of solving the problems of *Min-p-center* and *Max-p-center* in polynomial time (while $P \neq NP$). So, we focus on approximation solutions.

3.1. Max- p-center Problem

In this part, we consider the *Max-p-center* problem when the regions of uncertainty are modeled as (i) disjoint disks, or (ii) discrete sets. We present a $2-$approximation algorithm for the disjoint disks and extend it to provide a parameterized approximation algorithm for the special case of the Max-p-center problem when the regions are *well-separable* (see Definition 1).

Our algorithm is very simple, that is, "Choose the centers of the regions of uncertainty as the output instance". For the discrete sets, it is sufficient to choose the points whose maximum distance from the other points of the set is minimized. In Theorem 2, we show that such placement results in a $2-$approximation solution. Note that, the outcome of this algorithm is an instance, i.e., an approximation for I^{max}, and to find a solution for the p-center problem of such instance, we can use a simple iterative 2-approximation greedy approach [29].

Through this paper, we point out an assignment of the demand points to the facility centers as a *clustering* process. In fact, the difficulty of the p-center problem is how to cluster the demand points, and if it is known, the optimal location of the centers can be obtained in linear time. An example of clustering with three clusters is shown in Figure 1. In a cluster, each demand point is served by its nearest center, so, it can be seen as a graph of p stars. The edges of this graph are between the demand points and their corresponding nearest center, and we refer to the distance between them as the *weight* of that edge. Two clusters have the same *structure* if the assignments of demand points to the centers are the same. In other words, for any pair of demand points, if they

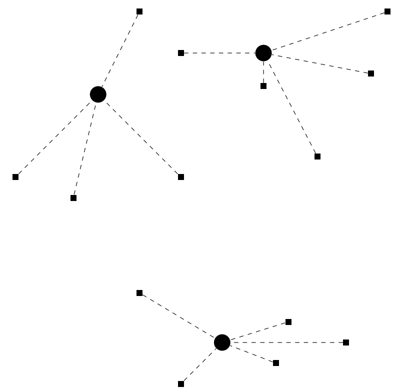


Figure 1: Clustering for three facility centers.

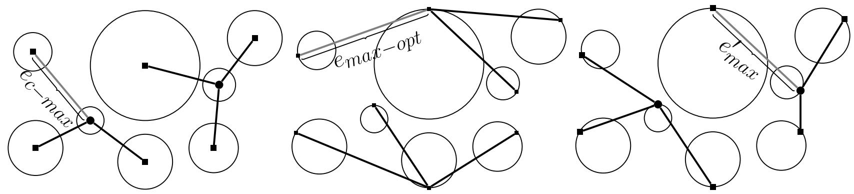
are served by the same center in one cluster, they are served by the same center in the other cluster as well.

Theorem 2. *Let D be a set of disjoint disks as the regions of the uncertainty in the Max-p-center problem. The algorithm that chooses the center of disks as the output instance is a $2-$approximation algorithm.*

Proof. We consider three clusters C_{opt}, C_{c-opt} and C'. C_{opt} is the solution of the Max-p-center problem, e.g., I^{max}, C_{c-opt} is the optimal solution when the centers of the disks are chosen, and C' is the cluster which has the same structure with C_{c-opt} and the same placement with I^{max}. We compare C_{c-opt} and C_{opt} using C'. In the p-center problem, the goal is to minimize the maximum length edge in all the clusters. Let $e_{max-opt}$ be the maximum distance between any demand point and its assigned center in C_{opt}. Actually, $e_{max-opt}$ is the edge with maximum length in C_{opt}. Similarly, let e_{c-max} and e'_{max} be the edges with maximum length in C_{c-opt} and C', respectively. Figure 2 illustrates the clusters C_{c-opt}, C_{opt} and C'. Since the location of demand points in C' and C_{opt} are the same, thus

$$e_{max-opt} \leq e'_{max}. \tag{1}$$

Since C_{c-opt} and C' have the same structure, if the location of the demand points changes anywhere on disks, the length of each edge increases at most as



(a) C_{c-opt} (b) C_{opt} (c) C'

Figure 2: Three different clusters for the Max-p-center problem.

much as the sum of the radius of two (disjoint) disks. So,

$$e'_{max} \leq 2e_{c-max}. \tag{2}$$

According to inequalities 1 and 2, we have

$$e_{max-opt} \leq 2e_{c-max}. \tag{3}$$

We compare the corresponding edges in C_{c-opt} and C'. Note that, the longest edges in these two clusters may be different. We claim that inequality 2 is established even for this case. Suppose that in C_{c-opt}, e is corresponding edge with e'_{max} in C'. So,

$$e'_{max} \leq 2e. \tag{4}$$

Since e_{c-max} is edge with maximum length and e is an edge in C_{c-opt}, then

$$e \leq e_{c-max}. \tag{5}$$

According to inequalities 4 and 5

$$e'_{max} \leq 2e_{c-max}. \tag{6}$$

Therefore the inequality 3 holds even for this case. Consequently, the proof is complete. □

Theorem 2 states that the set of the center of disks constructs a $2-$approximation for I^{max} when the disks are disjoint. In the following, we show that there is a nice relationship between the approximation ratio of such a solution and *separability factor* of the disks by proposing a parameterized approximation ratio.

Definition 1. For a set of disks D, let r_{max} be the radius of the largest disk. D is called *k-separable*, if the minimum distance between any pair of disks in D is at least $k \cdot r_{max}$. For an input such as D, *separability* is the maximum k such that D is k-separable.

Theorem 3. *Let D be a set of k-separable disks as the region of uncertainty in the Max-p-center problem. The algorithm that places the center of disks as the instances of the demand points is a $(1 + \frac{2}{k+2})-$approximation algorithm.*

Proof. This proof is similar to the proof of Theorem 2. We consider C_{c-opt}, C' and C_{opt} as before. Suppose e' is an arbitrary edge in C', and d_i and d_j are two disks connecting with e'. Let r_i and r_j be the radius of d_i and d_j, respectively, and l be the distance between d_i and d_j. Also, let e be the corresponding edge with e' in C_{c-opt} whose weight is $l + r_i + r_j$. The weight of e' is at most $l + 2r_i + 2r_j$. So, the weight of an edge in C' to the weight of its corresponding edge in C_{c-opt} is at least:

$$\frac{e'}{e} = \frac{l + 2r_i + 2r_j}{l + r_i + r_j} \leq \frac{k \cdot r_{max} + 2r_i + 2r_j}{k \cdot r_{max} + r_i + r_j}$$

$$\leq \frac{k \cdot r_{max} + 2r_{max} + 2r_{max}}{k \cdot r_{max} + r_{max} + r_{max}} = \frac{k+4}{k+2}. \tag{7}$$

This inequality holds for any edge in C_{c-opt}. So, regarding the inequality 7,

$$e'_{max} \leq \frac{k+4}{k+2}\ e_{c-max}, \tag{8}$$

where e_{c-max} is the edge with maximum weight in C_{c-opt} and e'_{max} is the edge with maximum weight in C'. Since C_{opt} and C' have the same demand points, so,

$$e_{max-opt} \leq e'_{max}, \tag{9}$$

where $e_{max-opt}$ is the edge with maximum weight in C_{opt}. According to inequalities 8 and 9, we have

$$e_{max-opt} \leq \frac{k+4}{k+2}\ e_{c-max}. \tag{10}$$

Therefore, the set of center of the disks is $\frac{k+4}{k+2} = (1 + \frac{2}{k+2})-$ approximation solution. □

Now, we show that the idea behind the parameterized approximation algorithm can be applied to the region of uncertainty of the demand regions modeled by discrete sets. We are given a set of points for each uncertainty region, e.g., $S = \{S_1, S_2, ..., S_n\}$, where S_i, for $i = 1, 2, ..., n$, is a set of points. The goal is finding an instance I, such that $p - center(I)$ is maximized. Similarly, S is called *k-separable*, if the minimum distance between any pair of regions is not less than k times the maximum distance between the points in any region.

Theorem 4. *Let $S = \{S_1, S_2, ..., S_n\}$ be a set of k-separable regions of uncertainty which are modeled by discrete sets. The Max-p-center problem for S can be solved with $\frac{k+4}{k+2} = (1 + \frac{2}{k+2})$ approximation ratio.*

Proof. The proof is similar to the proof of Theorem 3. It is sufficient that instead of choosing the center of disks as the instances, choose the point $s_i \in S_i$ whose maximum distance from any point in S_i is minimized. The point s_i is the solution of the discrete 1-center problem [8], that is, the smallest enclosing circle of S_i whose center should be one of the points in S_i. Since the solution of 1-center for each set S_i is a point of S_i whose maximum distance from the other points in S_i is minimum one, it satisfies the necessary conditions for the proof. □

3.2. Min-p-center Problem

In this subsection, we study the Min-p-center problem. As defined in the previous section, the goal of the problem is finding an instance I among all possible instances, such that $p - center(I)$ is minimized. Let I^{min} denote such an instance. Similar to the Max-p-center problem, we show that choosing the center of each region results in a good approximation for I^{min}, i.e., a $(1 + \frac{2}{k})-$approximation solution when the regions are k-separable.

Theorem 5. *Let D be a set of k-separable disks as the regions of the uncertainty in the Min-p-center problem. The algorithm that places the center of disks as the instances of the demand points is a $(1+\frac{2}{k})-$ approximation algorithm.*

Proof. This proof is similar to the proof of Theorem 3, however, the definition of the clusters is different. Let C_{opt} be the solution of Min-p-center problem, C_{c-opt} be the solution of p-center for the center of the disks and C' be the cluster which has the same structure with C_{opt} and the same location of demand points with C_{c-opt}. Since both C_{c-opt} and C' are the clusters on the center of disks and C_{c-opt} is the optimal solution of the p-center problem, we have

$$e_{c-max} \leq e'_{max}, \tag{11}$$

where e_{c-max} is the edge with maximum weight in C_{c-opt} and e'_{max} is the edge with maximum weight in C'.

We consider an arbitrary edge $e' \in C'$. Suppose d_i and d_j are two connecting disks by e'. Let r_i and r_j be the radius of d_i and d_j, respectively, and l be the maximum distance between d_i and d_j. Suppose two disks d_i and d_j in C_{opt} are connected by an edge e whose weight is at least l. So, the weight of e' is at most $l + r_i + r_j$. So, the weight of an edge in C_{opt} to the weight of its corresponding edge in C' is at least

$$\frac{e}{e'} = \frac{l}{l + r_i + r_j} \geq \frac{k \cdot r_{max}}{k \cdot r_{max} + r_i + r_j}$$

$$\geq \frac{k \cdot r_{max}}{k \cdot r_{max} + r_{max} + r_{max}} = \frac{k}{k+2}. \tag{12}$$

This is established for any edge in C_{opt} and its corresponding edge in C'. So,

$$e_{max-opt} \geq \frac{k}{k+2} e'_{max}, \tag{13}$$

where $e_{max-opt}$ is the edge in C_{opt} with maximum length. According to inequalities 11 and 13, we have

$$e_{max-opt} \geq \frac{k}{k+2} e_{c-max}. \tag{14}$$

Thus, the proof is complete. $\square$

Theorem 6. *The problem Min-p-center for a set of $k-$separable discrete sets as the regions of the uncertainty can be solved with $\frac{k+2}{k} = (1 + \frac{2}{k})-$ approximation ratio.*

Proof. Similar to the proof of Theorem 4 and Theorem 5. It is sufficient to choose the solution of the discrete 1-center problem [8] as the instance I^{min}, and follow the proof of Theorem 5. $\square$

4. MinMax Regret 1-center problem

In this section, we study the planar MinMax regret 1-center problem under the Manhattan and Euclidean metrics. As aforementioned, general cases of the MinMag regret are NP-hard and only special cases of MinMax regret 1-center problem have been solved in polynomial time. We assume a simple case of uncertain demand points where the regions of uncertainty are horizontal segments and present a linear-time algorithm for the Manhattan metric as well as an $O(n^4)$ time algorithm for the Euclidean metric.

Let $\Re = \{R_1, R_2, \ldots, R_n\}$ be a set of n regions of uncertainty in the plane as the n locational uncertain demand points, and $I = \{p_1, p_2, \ldots, p_n\}$ be an instance of it, i.e., $p_i \in R_i$, for $i = 1, 2, \ldots, n$. For a point $x \in \mathbb{R}^2$, $F(x, I)$ is defined as follows:

$$F(x, I) = \max_{1 \leq i \leq n} d(x, p_i), \tag{15}$$

where $d(x, p_i)$ is the distance (Manhattan or Euclidean in this paper) between x and p_i. The optimal solution of 1-center problem for the instance I can be defined as follows

$$F^*(I) = \min_{x \in \mathbb{R}^2} F(x, I). \tag{16}$$

Now, the difference value $F(x, I) - F^*(I)$ is called *Regret* for a point x and an instance I. Let denote the worst case of the regret for x by *MaxREGR* defined as follows

$$MaxREGR(x) = \max_{I \in \Omega} \ (F(x, I) - F^*(I)), \tag{17}$$

where Ω is the set of all possible instances. The MinMax regret 1-center problem is finding x such that MaxREGR(x) is minimized. MinMax regret solutions are sometimes called *Robust* solution [30] as well. For the sake of simplicity, we denote this problem by ROB which is

$$ROB(\Re) = \min_{x \in \mathbb{R}^2} \quad \max_{I \in \Omega}(F(x, I) - F^*(I)).$$

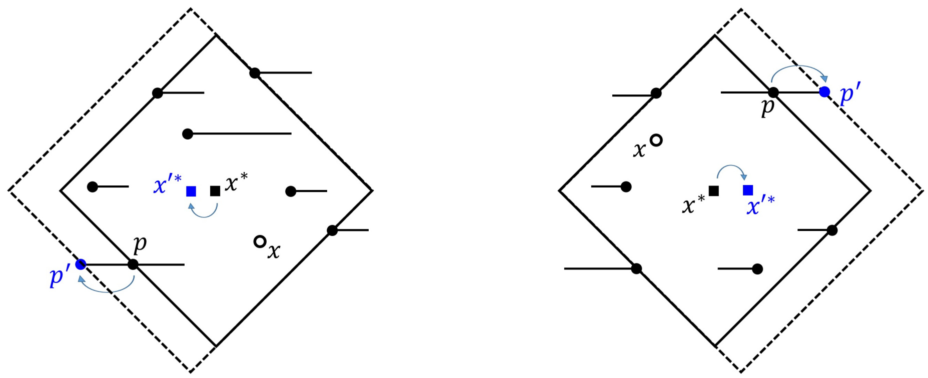
We consider both the Manhattan and Euclidean metrics and propose algorithms for ROB where $\Re$ is a set of horizontal segments.

4.1. MinMax Regret 1-center Problem under the Manhattan Metric

In this section, we present a linear-time algorithm for the MinMax regret 1-center problem under the Manhattan metric, where the regions of uncertainty are horizontal segments. Let $I_r \in \Omega$ and $I_l \in \Omega$ denote the two particular instances including the rightmost and the leftmost placements of the segments, respectively. Also, let p_l and p_r be the solution of 1-center problem for the instances I_r and I_l, respectively. From the geometric point of view, the optimal solution of the 1-center problem under the Manhattan metric is the smallest square which is rotated $\frac{\pi}{4}$ and contains all instances of the demand points. To determine such a square, we need at most four boundary points (two in the degenerated case). Let call these boundary points *critical* points [31].

Lemma 1. *For any point $x \in \mathbb{R}^2$, two instances which lead to the $MaxREGR(x)$ in the Manhattan metric are either I_r or I_l when the regions of uncertainty are horizontal segments.*

Proof. Clearly, we only need to consider the critical points (segments) that determine the smallest enclosing $\frac{\pi}{4}$ rotated square, and for the segments that lie completely inside the square, we can freely move their chosen points to the left or right endpoints. Assume to the contrary there is an instance $I \in \Omega$ different from I_r and I_l such that leads to $MaxREGR(x)$ for $\mathbb{R}^2$. So, $F(x, I) - F^*(I)$ is the maximum difference value among all possible instances. Now, let p be the one of points on the boundary of the rotated square of 1-center solution



(a) The case x lies right side of the optimal 1-center solution of I

(b) The case x lies left side of the optimal 1-center solution of I

Figure 3: Constructing a worse instance I' using an instance I whose some chosen point, like p, does not belong to the endpoints of the segments. It is sufficient to move p to the left endpoint (a) or to the right endpoint (b) to obtain a larger value for ROB.

in the Manhattan metric such that has the maximum distance from x, i.e., $d(x, p) = F(x, I)$. Let x^* be the solution of the 1-center problem for the instance I. Thus, $F^*(I) = d(p, x^*)$. Thus, $F(x, I) - F^*(I) = d(x, x^*)$. If x^* is the left (right) of the x, we can construct a new instance $I' \in \Omega$ by moving p to the left (right) endpoint of its corresponding uncertainty region results. Since we only move p in the horizontal direction, the solution of 1-center problem for the instance I' compared to x^* moves in the same direction as well. Figure 3 displays an example for this case. Let denote the optimal solution for I' by x'^*. Observe that $d(x, x^*) \leq d(x, x'^*)$ and it means $F(x, I) - F^*(I) \leq F(x, I') - F^*(I')$. Therefore, I does not lead to the $MaxREGR(x)$, which is a contradiction. □

This lemma results in an efficient approach to computing the optimal solution for the ROB problem under the Manhattan metric by considering the critical leftmost and rightmost instances.

Theorem 7. *The ROB problem under the Manhattan metric can be computed in linear time when the uncertainty regions of the demand points are horizontal segments.*

Proof. According to Lemma 1, either I_r or I_l leads to $MaxREGR(x)$. So, consider x_r^* and x_l^* are the solutions of 1-center for I_r and I_l, respectively. So, the middle point of x_r^* and x_l^* is the solution to the ROB problem. Since such solutions can be computed in linear time, so does the ROB problem. □

4.2. MinMax Regret 1*-center Problem under the Euclidean Metric*

In this section, we study the ROB problem under the Euclidean metric, where the regions of uncertainty are horizontal segments. From the geometric point of view, the Euclidean 1-center problem is as finding the minimum circle covering all the demand points. Megiddo proposed a parametric search algorithm for this problem in linear time [9]. We utilize this algorithm to solve the ROB problem. First, we show that the endpoints of the segments play the main role in determining the optimal solution of ROB.

Lemma 2. *When the location of demand points are horizontal segments, for any point $x \in \mathbb{R}^2$, $MaxREGR(x)$ is a distance between x and some endpoints of the segments.*

Proof. Assume to the contrary $I \in \Omega$ is an instance that leads to the $MaxREGR(x)$ and includes some boundary placement, such as $p \in I$ that does not belong to the endpoints of its corresponding segment. Similar to the Manhattan case, it is possible to construct a worse instance by moving p toward one of the left or right endpoints. Let $p \in I$ lie on the boundary of the minimum covering circle of I, so, x has the maximum distance from p compared to other points of I. Thus, $d(p, x) = F(x, I)$ and $F^*(I) = d(p, x^*)$, where x^* is the center of I. If x^* is the right (left) side of x, then by moving p to the right (left) endpoint (denote by p'), a worse instance $I' \in \Omega$ be constructed. Let denote the center of the minimum covering circle of I' by x'^*. Observe that $d(p, x) \leq d(p', x)$. Such a movement results in the center of the minimum covering circle moves to the right (left) as well. Thus, $d(p, x^*) \leq d(p, x'^*)$. We have $d(p', x) - d(p, x) \geq d(p, x'^*) - d(p, x^*)$. So, $F(x, I) - F^*(I) \leq F(x, I') - F^*(I')$. Therefore, I cannot be a solution for $MaxREGR(x)$, which is a contradiction. □

Lemma 2 helps to discretize the search space of the ROB problem and confine it to only the endpoints of the regions of the uncertainty (the demand segments). However, there is an exponential number of combinations of the endpoints and we need to prune the combinations that do not affect the optimal solution of the problem.

Lemma 3. *For a set of n regions of uncertainty whose shapes are horizontal segments, there are at most $O(n^3)$ different instances as the possible candidates for the optimal solution of the ROB problem.*

Proof. As a simple fact, among a set of points in the plane, only two or three points determine the minimum enclosing circle of the points. So, at most $O(n^3)$ triple of points may be considered. On the other hand, lemma 2 showed that the optimal solution of the ROB problem is obtained from the instances in which endpoints of segments are chosen. So, there are $O(n^2)$ different pairs and $O(n^3)$ different triples of the segments that should be investigated. Also, for each triple, there are $2^3 = 8$ combinations of the endpoints. $\square$

Therefore, there are $O(n^3)$ candidates for the optimal solution of the ROB problem. Note that, we need to consider only the combinations that their minimum circle covers all the demand points. Since such a feasibility check can be done in linear time, the whole process of finding all feasible candidates takes $O(n^4)$ time.

Theorem 8. *For a given set of n regions of uncertainty whose shapes are horizontal segments, the ROB problem under the Euclidean metric can be solved in $O(n^4)$ time.*

Proof. According to Lemma 3, there are at most $O(n^3)$ candidate instances for the ROB problem and their feasibility can be verified in linear time. Thus, in $O(n^4)$ it is possible to compute all feasible centers of the minimum covering circles of the demand points. Each center can play an optimal solution for the Euclidean 1-center problem for some instances of the demand points. Regarding the definition of the ROB problem, we need to find a point whose maximum difference from such optimal centers is minimized. That is, again we should compute the minimum covering circle of such centers. Since there are at most $O(n^3)$ centers, using a linear time approach such as Megiddo's algorithm [9], the centroid of the minimum covering circle of these centers can be computed in an additional time $O(n^3)$. Consequently, the ROB problem can be solved in $O(n^4)$ time. $\square$

5. Conclusion and Future Work

The p-center problem is a well-known facility location problem with significant real-world applications and has been studied considerably. Since such applications may encounter data uncertainty, we studied this problem in the context of uncertainty, that is, the location of the demand points may change in a predefined shape such as disk, segment, or discrete sets. We introduced three different versions of the problem, called *Max-p-center problem*, *Min-p-center problem* and *MinMax Regret* 1-*center*, and proposed approximation and polynomial time algorithms to solve them. All three problems are NP-hard, and we present $2-$approximation solutions as well as parametrized algorithms for the first and second problems. Our algorithms work only for disk-shaped regions and discrete sets which are well-separated. Further, for the third problem, we only considered the problem for the case $p = 1$, and proposed a linear time for the Manhattan metric as well as an $O(n^4)$ time algorithm for the Euclidean metric, where n is the number of demand points. In this problem, we assumed horizontal segment-shaped regions.

Since p-center problem in the general case, when p is a part of the input, is an NP-hard problem, its extensions to the uncertain demand points remain NP-hard. Here, we assumed uncertainty only for the location of the demand points, however, it can be generalized for the defined cost function between the centers and the demand points. Further, we assumed the uncertainty regions for the disk-shaped and discrete sets. As an interesting future direction, it may consider general shapes of regions of uncertainty. Also, for the MinMax-Regret problem, it is interesting to find solutions for $p > 1$.

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